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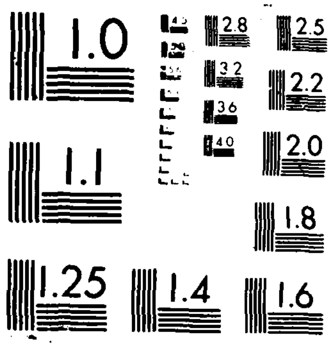
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U.S. Army Intelligence Center and School Software Analysis and Management System

Two Dimensional Uncorrelated Bias In Fix Algorithms

Technical Memorandum No. 20

Mathematical Analysis Research Corporation

EXTENDED EAAF ALGORITHM ANALYSIS FAMILY

10 June 1987

National Aeronautics and
Space Administration

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"Bias" in Location Estimation Algorithms is defined to be a vector from the "estimated" fix site to the "true" Fix site. Four Fix Algorithms are examined in this report: Least Squares (Stanfield Approximation), Weighted Least Squares, Minimization of Angular Error, and minimization of the Size of the Angular Error. Analysis expressions for the First-Order error are derived for two specific cases: Two Lines-of-Bearing (LOB's) and infinite LOB's. The analytic derivations are relegated to appendices. Three questions are answered by this report. First, which algorithms cause the largest bias? Second, how

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sensitive is this bias to sample size, system noise, and variance? Third, which algorithms (if any) are asymptotically unbiased? All conculsions are then verified by simulation.

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Software Analysis and Management System

Two Dimensional Uncorrelated Bias In Fix Algorithms

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PREFACE

The work described in this publication was performed by the Mathematical Analysis Research Corporation (MARC) under contract to the Jet Propulsion Laboratory, an operating division of the California Institute of Technology. This activity is sponsored by the Jet Propulsion Laboratory under contract NAS7-918, RE182, A187 with the National Aeronautics and Space Administration, for the United States Army Intelligence Center and School.

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TWO DIMENSIONAL UNCORRELATED BIAS IN FIX ALGORITHMS

NO. 125

TABLE OF CONTENTS

I. INTRODUCTION

The purpose of this report is to analyze the various fix algorithms used in determining estimates of emitter location (fixes) based on varying numbers of Lines of Bearing (LOBs). Since sensors do not produce infallible readings, observations (LOBs on an emitter) are subject to measurement error and the actual estimated location will differ from the true location. The difference between the expected value of the estimates and the true emitter location is called bias[1]. This analysis focuses on how the bias of the different algorithms is affected by increasing or decreasing sample size (number of LOBs), varying sensor error, and changing the relative locations of emitters and sensors.

From observations, it is possible to calculate an estimate of the true location of the emitter by utilizing different fix algorithms. Four 2-dimensional algorithms were chosen for study in this report: the Perpendicular, the Weighted Perpendicular, the Sine of Error Minimization, and the Minimization of Angular Error methods. These algorithms were chosen because they represent variations on classical fix theory and because they relate to fix methods of actual systems. For instance, the Perpendicular method is the 'classical' method introduced by Stansfield; the Weighted Perpendicular method is similar to the fix algorithm in Guardrail V; the Sine of Error Minimization method is comparable to that used in FFIX; and the Minimization of Angular Error method is the theoretically 'ideal' approach according to some sources. Although they are indeed variations, the four methods presented here are not altogether different from each other.

In general, each of the four fix methods mentioned above is biased. General expressions (located in the appendix) for first-order bias[2] have been derived for each of the methods. In their general form, these expressions can be used for calculations but are too complicated to interpret qualitatively. So, the qualitative analysis of bias has been divided into intermediate steps. Since the bias of the methods changes as a function of the number of LOBs, bias has first been analyzed at the extreme bounds of sample size: 2 LOBs and infinite LOBs. Then, special cases have been examined between these bounds to obtain a better intuitive grasp for the behavior of bias.

Conclusions drawn from these expressions are supported by simulation results and will be illustrated by diagrams. It is the goal of this report to examine certain pertinent questions regarding bias. Which situations or which fix methods produce the most bias? How sensitive is bias to sample size and other factors? Answers to these and similar questions are discussed in the following text.

[1] Formally, let $(\hat{x},\hat{y})$ be the estimated location, based on a particular algorithm, and (x_0,y_0) be the true location. Then, bias is defined to be the vector from (x_0,y_0) to the expected value of $(\hat{x},\hat{y})$.

[2] Bias has been expanded using a multivariate Taylor Series expansion. The higher order terms, which are generally insignificant, have been dropped. See the appendix for more detail.

II. CONCLUSION

The particular properties for the various methods in special cases are discussed in depth in section III. Here, however, broad results concerning bias are summarized. First, results pertaining to the behavior of bias in general are outlined. Then, the way in which bias behaves as a function of sample size has been carefully described. For detailed descriptions of the various methods and how they operate, see the appendix.

For the purposes of this report, bias has been defined to be a vector. As such, it has a magnitude and a direction. Both the magnitude and the direction are affected by sample size and the configuration of the sensors. In addition, however, the magnitude is also affected by the standard deviation of the angular errors.

The standard deviation (σ) associated with the sensor measurement error will affect the size of bias, regardless of the method, the configuration, or the sample size. The magnitude of bias is proportional to σ^2. In Figures 1,2,and 3, the large region surrounding the emitter is an indication of the spread of the estimates while the smaller region in contact with the emitter represents bias. Notice that when σ has been doubled, the region of bias has become four times larger. In Figure 3, σ has been tripled and, subsequently, the region of bias has increased by a factor of nine. Thus, the extent to which bias can be a problem clearly depends heavily on the size of σ.

Another major factor in determining the magnitude of bias is the configuration of the sensors. The simplest situation which will produce large bias is any configuration in which the angles between the true lines of bearings are small. Whenever these angles are less than 30 degrees, the bias behavior is such that if the angle is decreased by a factor of n, then the bias will increase by a factor of n^2. So, if the angle is decreased from 12 degrees to 6 degrees, the bias will increase by a factor of four. Bias can also be a problem if the angles between the true lines of bearings are sufficiently close to 180 degrees, although these angles will not generally cause as severe a problem as the thin angles.

An illustration of a situation in which the magnitude of bias is large is depicted in Figures 4 and 5. In the first diagram, σ is small, and yet bias is still rather large due to the thinness of the angles between bearings. The next diagram clearly shows the impact of σ. The already large bias has been increased by a factor of nine, making it enormous, due to the tripling of σ.

Note that increasing σ by a factor of two only increases the lengths of the major and minor axes of the error ellipse[1] by a factor of two. The bias, on the other hand, increases by a factor of four. At some point, bias will actually reside outside the ellipse. The relative importance of random error represented by both the ellipse and bias changes dramatically with changes in σ.

In the general cases under study, however, bias appeared to be relatively small. As long as the angles between true lines of bearings are not that small, then, even if σ equals 3 degrees, bias will not be very large. See Figure 6. With σ=2 degrees, the "bias region" is much, much smaller than the "spread region" ABCD for this typical configuration. On the other hand, if the angles between the bearings are extremely thin, then bias can be large even with σ=1 degree (as in Figure 4).

[1] The error ellipse is a confidence ellipse which contains the emitter with probability (confidence) β. β is usually chosen to be .95

A. Boundary Cases -- Sample Size

As mentioned before, the general expressions for first-order bias defy simple qualitative characterization. In their general form, the expressions contain many factors which contribute to bias behavior. Besides the two crucial factors previously mentioned, the standard deviation and the angle between LOBs, the next most significant factor which contributes to bias is sample size. So, one approach of analysis has been to examine the behavior of bias at the extreme ends of sample size: the 2 LOB case and the infinite LOB case.

1. 2 LOB Case

In the 2 LOB case, all of the fix algorithms will produce the same estimate: simply the intersection of the 2 LOBs. Refer to Figure 7. The direction of bias is determined by the relative location of the emitter to the sensors. If the emitter is outside the circle shown, then bias will point away from the sensors; if the emitter is on the circle, first-order bias equals zero; and if the emitter lies within the circle, then bias will point towards the sensors. Generally, however, bias has been analyzed in terms of the angle between the true bearings. The size of this angle is directly correlated to the emitter's position relative to the circle. When the emitter is outside the circle, the angle will be less than 90 degrees; inside the circle, greater than 90 degrees; and on the circle, the angle between the true bearings will equal 90 degrees.

2. Infinite LOB Case

In the infinite LOB case, bias is dependent on the fix algorithm applied. Two of the methods, the Minimization of Angular Error and the Sine of Error Minimization, will produce no first-order bias in their estimates as sample size approaches infinity (asymptotically first-order unbiased), while the other two methods, Perpendicular and Weighted Perpendicular, are not unbiased as sample size approaches infinity.

It is worthy of noting the relative importance of bias, as opposed to its absolute size, in the 2 LOB case as compared to the infinite LOB case. For the Weighted Perpendicular and the Perpendicular methods, the error in the estimate for the 2 LOB case (or any case with small sample size) consists mostly of noise and bias is a relatively small percentage of the error. As sample size approaches infinity, however, the noise shrinks (corresponding to the shrinking of confidence ellipses), and the error in the estimate is composed almost entirely of bias. Also, the expected location of the infinite LOB estimate is 'short of the true' for these two methods.

For the Minimization of Angular Error and the Sine of Error Minimization methods, the relative importance of bias does not increase as sample size increases. Rather, the relative importance decreases because the bias shrinks faster than the confidence ellipse as LOBs are added.

B. Intermediate Cases

When the number of LOBs is greater than 2, the "expected location" and hence bias lies between the 2 LOB and the infinite LOB cases (between can mean zero bias, however). At both the 2 LOB and infinite LOB cases, bias has already been well-characterized. In between these boundaries, the important

aspect to note about the properties of bias is the movement of bias between the two extreme cases. This movement betwee the extremes is also well-characterized if the configuration of the sensors is maintained and if more readings are simply taken proportionately from each sensor. In general, however, in order to characterize bias in between these boundaries, it was necessary to restrict attention to particular configurations of emitters and sensors. These particular results for the special configurations are discussed in the next section.

III. PROPERTIES OF THE GENERAL FORMULAS

In this section, various results regarding bias are listed according to particular cases. Mathematical formulas , derivations, and proofs are contained in the Mathematical Appendix.

(Many of the results cited in this section have been described under the assumption that the sensors are only located on one side of the emitter)

Four general 2-dimensional fix algorithms were analyzed for the purpose of this report. Three of these algorithms (all but the Weighted Perpendicular) can be defined as Least Squares Fix Algorithms, which are algorithms that estimate parameters by minimizing a sum of squared terms (usually a measure of error). A general formula for least squares algorithms has been derived (located in the appendix) which is applicable to all three methods. The particular formulas then only differ by the particular term that is to be squared and summed. For example, the Perpendicular Method uses the perpendicular distance from the estimate to the LOB as the term to be squared while the Minimization of Angular Error Method uses the angle between the LOB and the line from the sensor to the location estimate as the term to be squared. Since the formula applies to all three least squares algorithms, any analysis of the formula will be relevant to all three methods. For example, the infinite LOB case has been simply characterized and also the movement of bias as sample size increases from 2 to infinity by using the general formula.

The results regarding bias in this section of the report only refer to first-order bias and not to total bias. When the standard deviation is small, however, the higher orders of the bias term do not contribute very much and first-order bias is a close approximation of total bias. Simulation results have been used to verify that this approximation is reasonable.

In the rest of this section, the properties of each of the algorithms except for the Weighted Perpendicular method are described in depth. For the Weighted Perpendicular method, there is an extra complication. Relative distance from the sensors to the emitter has an effect. Except for this factor, the Weighted Perpendicular method is identical to the Perpendicular method. In general, the Weighted Perpendicular method emphasizes the data from the closest sensors while the Perpendicular method averages the data from all the sensors. Thus, conceptually, the Weighted Perpendicular behaves like the Perpendicular "for the closest sensors".

First-order bias for the Sine of Error Minimization method is identical to that given by the Minimization of Angular Error method. So, properties regarding first-order bias for the Minimization of Angular Error method can be assumed also to hold for the Sine of Error Minimization method.

A. Extreme Cases -- Sample Size

1. 2 LOB Case

In the case when there are only 2 LOBs, the four fix algorithms will produce identical estimates. The estimate is simply the intersection of the 2 LOBs. The general scenario of the 2 LOB case is depicted in Figure 8. The true bearings are shown going through the emitter with corresponding angles θ_1 and θ_2 while ε_1 and ε_2 are the angular errors associated with each sensor. With the natural assumption of symmetry for angular error (errors of ε and $-\varepsilon$ are equally likely -- this assumption is used throughout this report), the

region ABCD may be constructed as shown. LOBs taken in this situation with angular errors smaller than the bounds ε_1 and ε_2 will intersect within this region and consequently, ABCD is a geometric representation of the spread of all possible estimates (with angular errors smaller than the bound).

Bias, however, is much smaller. Using the previous assumption of symmetry for angular error, the "region of bias" formed by cancellation of symmetric error in ABCD is shown back in Figure 6. Furthermore, it is possible to show that this "bias region" resulting from cancellation of symmetric error will always be wholly situated on one side of the emitter, depending on the size of the angle between the true bearings. Refer back to Figure 3. The angle between the true bearings is less than 90 degrees and thus the bias region is located entirely on the far side of the emitter. If the angle is greater than 90 degrees, as in Figure 9, then the bias region will be entirely on the close side of the emitter.

2. Infinite LOB Case

In the case of an infinite number of LOBs, three of the methods can be easily analyzed using the least squares general formula. This analysis, however, assumes that the configuration of the sensors is not changed significantly as sample size goes to infinity. If LOBs are added from sensors so that the total geometry of the situation is indeed changed significantly, then the behavior of bias is not as easily predicted (remember, the configuration can have a major effect on both the direction and magnitude of bias). The analysis for the least squares methods is as follows.

Least Squares Methods

As sample size increases, first-order bias for the Minimization of Angular Error (and Sine of Error Minimization) method approaches zero. In Figures 10-11, a comparison is made between a particular configuration using 2 LOBs and then 10 LOBs. As the diagram illustrates, the increase in sample size caused bias to shrink (also the spread of the fixes decreased). The speed at which the bias will shrink can be affected by significant changes in the configuration as LOBs are added (including changes in proportion of LOBs coming from a certain location). If, as in the diagram, there are no changes in the configuration, then the amount by which bias has shrunk is proportional to 1/n (n being the factor by which the sample size increased).

In general, asymptotic first-order bias for the Perpendicular method can be characterized in terms of the location of the emitter relative to the average sensor location. Refer to Figure 12. The heavy dotted line through the emitter is perpendicular to the thin dotted line from the average sensor location to the emitter. As sample size approaches infinity, the expected value of the estimated location of the emitter will lie short of the heavy dotted line. Without oversimplifying too much, one may summarize by saying that for sufficiently large sample size, one expects to be short of the true.

Weighted Perpendicular Method

The behavior of bias for the Weighted Perpendicular method from the 2 LOB to infinite LOB case need not be the same as the behavior of the other methods. Since it is not one of the least squares methods, it cannot be analyzed in the same manner as the others. However, in general, asymptotic first-order bias for the Weighted Perpendicular method will behave like that of the Perpendicular method.

In the special situation when there are only two symmetric sensors,

however, first-order bias is identical for the Perpendicular and the Weighted Perpendicular methods (this is explained on the next page, under "Symmetric Case") and so its behavior as sample size increases is also identical.

B. Transition -- Sample Size

The movement of bias in between the boundary cases for the three least squares methods can be determined by examining the general least squares formula. As the number of LOBs increases from 2 to infinity, the expected value of the estimated location will lie along the straight line from the 2 LOB estimate to the infinite LOB estimate (if there are no significant changes in the configuration). The amount by which the expected location differs from the infinite LOB estimate is proportional to $1/n$ (n being the factor by which the sample size increased) See Figure 13. When there are 2 LOBs, the expected value of the estimate is at Point A, and when there are an infinite number of LOBs, the expected value of the estimate is at Point B. As illustrated, the convergence is very fast; at only 4 LOBs, the expected value of the estimate is half the distance towards the infinite LOB estimate. If sample size is doubled again, the distance will be cut in half again. At only 16 LOBs, almost 90% of the distance to the infinite LOB estimate is already covered.

This same behavior will hold even if the starting sample size is not 2 LOBs. For instance, if LOBs are initially added to the first 2 LOBs such that the configuration is changed significantly, then the above analysis will not be valid. If the configuration does become stable at some point, so that all additional LOBs will not seriously affect the configuration, then the bias will begin to converge from that point towards the infinite LOB estimate. For example, assume that the first 4 LOBs have been attained such that each LOB changed the configuration and thus bias. Now, each additional LOB after the 4th LOB will be added without significant changes in the arrangement. Then, the expected value of the estimated location will now move in a straight line towards the infinite LOB estimate, moving half the remaining distance with each doubling of sample size. Thus, at 8 LOBs, the expected value of the estimate will be half the distance towards the infinite LOB estimate.

C. Variation of Bias With Position -- Special Cases

The following special cases have been included in order to provide certain insights to the behavior of bias.

1. Symmetric Case

The general symmetric scenario exists when the sensor configuration is symmetric about a line through the emitter; that is, the configuration on one side is a mirror image of that on the other. In addition, corresponding sensors on either side have been assumed to be equal in accuracy.

If it is assumed that corresponding sensors are equal in accuracy, then bias will lie entirely along the line of symmetry, regardless of the method. Thus, the symmetric assumption was chosen as the natural method of reducing bias to the determination of extent short and long.

Minimization of Angular Error Method

For the Minimization of Angular Error method, first-order bias in the symmetric situation is easily characterized. See Figures 14-15. If all of

the angles between the true bearings from symmetric sensors are less than 90 degrees, then bias will point long (along the line of symmetry) and if all of the angles are greater than 90 degrees and less than 180 degrees, then bias will point short. And if the angle equals 90 degrees, then first-order bias will equal zero.

Perpendicular Method

In a symmetric situation, Perpendicular bias has been defined with respect to the angle between true bearings and sample size. See Figures 16-19. When there are only 2 LOBs, the bias will point long along the line of symmetry when the angle between the true bearings is less than 90 degrees; it will point short when the angle is greater than 90 degrees; and the bias will be zero when the angle between the bearings equals 90 degrees (this behavior is consistent with that already shown for the general 2 LOB case).

As LOBs are added, both the direction and the magnitude of the bias vector can change depending on the size of the angle between the true lines of bearing. In Figures 16 and 17, all of these angles are greater than 120 degrees. Bias is already short using this configuration when there are only 2 LOBs; as sample size increases, bias will shrink and become less short (along the line of symmetry).

If all of the angles between true bearings are equal to 120 degrees, then the bias will <u>not</u> change in size or direction as sample size increases.

If these angles are less than 120 degrees (as in Figures 18 and 19), the bias vector will point short from whatever estimate it pointed to when there were only 2 LOBs. So, if the bias pointed long at 2 LOBs (angles less then 90 degrees), then it will change direction and point short. And if the bias pointed short at 2 LOBs (angles greater than 90 degrees but less then 120 degrees), then bias will point even shorter and increase in magnitude.

Weighted Perpendicular Method

When the relative distances from the sensors to the expected value of the estimated location are all the same, then the Weighted Perpendicular method will produce exactly the same first-order bias as the Perpendicular method. Thus, when all of the LOBs come from only two symmetric sensors, first-order bias will be identical for both of the methods.

2. <u>Effects of Range Differences Between Sensors</u>

How each method uses the data it receives has also been studied to some extent.

Perpendicular and Weighted Perpendicular Methods

As an example of how the Perpendicular and the Weighted Perpendicular methods can deviate in certain cases, a special configuration was used. In Figures 20-21, the sensors are placed on two concentric arcs so that sensors on the same arc are equidistant from the emitter. Furthermore, a sensor on one arc can be found placed exactly in line with the emitter and with a sensor on the other arc (so that the angle between the emitter and consecutive sensors are the same from one arc to the other). Thus, the only difference between the sensors on one arc from those on the other is the distance from the emitter.

As the figures illustrate, there is a marked difference between the

estimate that each method produces. The Weighted Perpendicular method has used the data from the closest sensors (and ignored the data from the farthest sensors) and has come up with a better estimate while the Perpendicular method has averaged the data from the sensors and has come up with a worse estimate.

Minimization of Angular Error

In an identical configuration as the one described above, the Minimization of Angular Error method will weight the data from the two concentric arcs so that only the data from the closest sensors is used and the data from the farthest sensors is virtually ignored.

3. Effects of Angles

Minimization of Angular Error

The general 90 degree case exists when there are only two sensor locations and the emitter is located such that the true bearings are 90 degrees apart (See Figure 22). This special case is a generalization of the 2 LOB, 90 degree case (illustrated in Figure 7), for which first-order bias equals zero. When multiple LOBs are taken from the sensors, first-order bias using the Minimization of Angular Error method equals zero, regardless of the respective number of LOBs taken from each sensor location or of the angular standard deviation associated with each sensor.

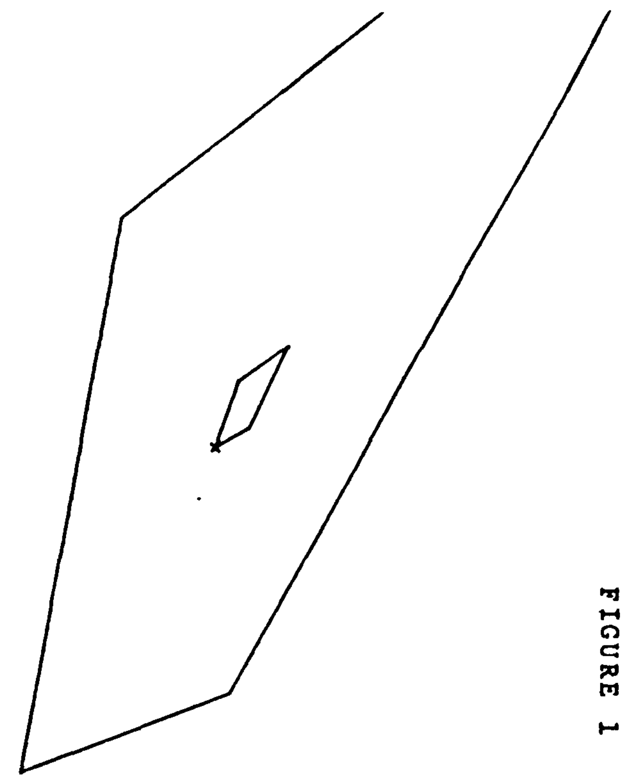
Perpendicular and Weighted Perpendicular Methods

The Perpendicular and Weighted Perpendicular methods are not unbiased when the true bearings are 90 degrees apart except when there are only 2 LOBs. For more than 2 LOBs, the bias for the Perpendicular method is not easily characterized because, unlike the Minimization of Angular Error method, its bias depends on the respective number of LOBs from each sensor and the angular standard deviation. As sample size approaches infinity, however, the estimate produced by either method will be biased short of the true location.

FIGURE 1

2 LOB ANALYSIS

σ=3 degrees



0 <--X--> 20

The outer region is bounded by 3σ.

The corresponding inner region is the bias region after cancellation of symmetric error.

FIGURE 2

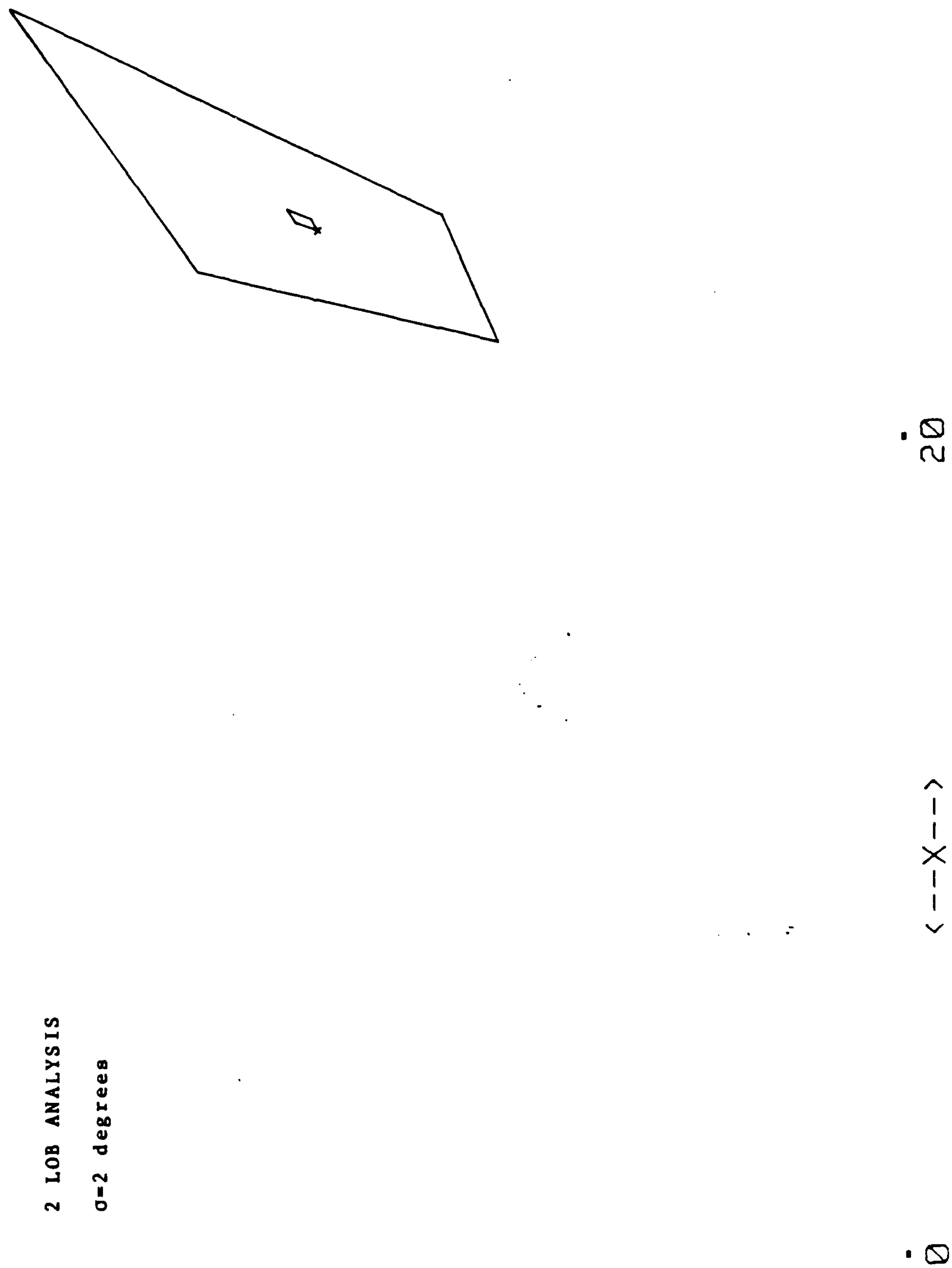
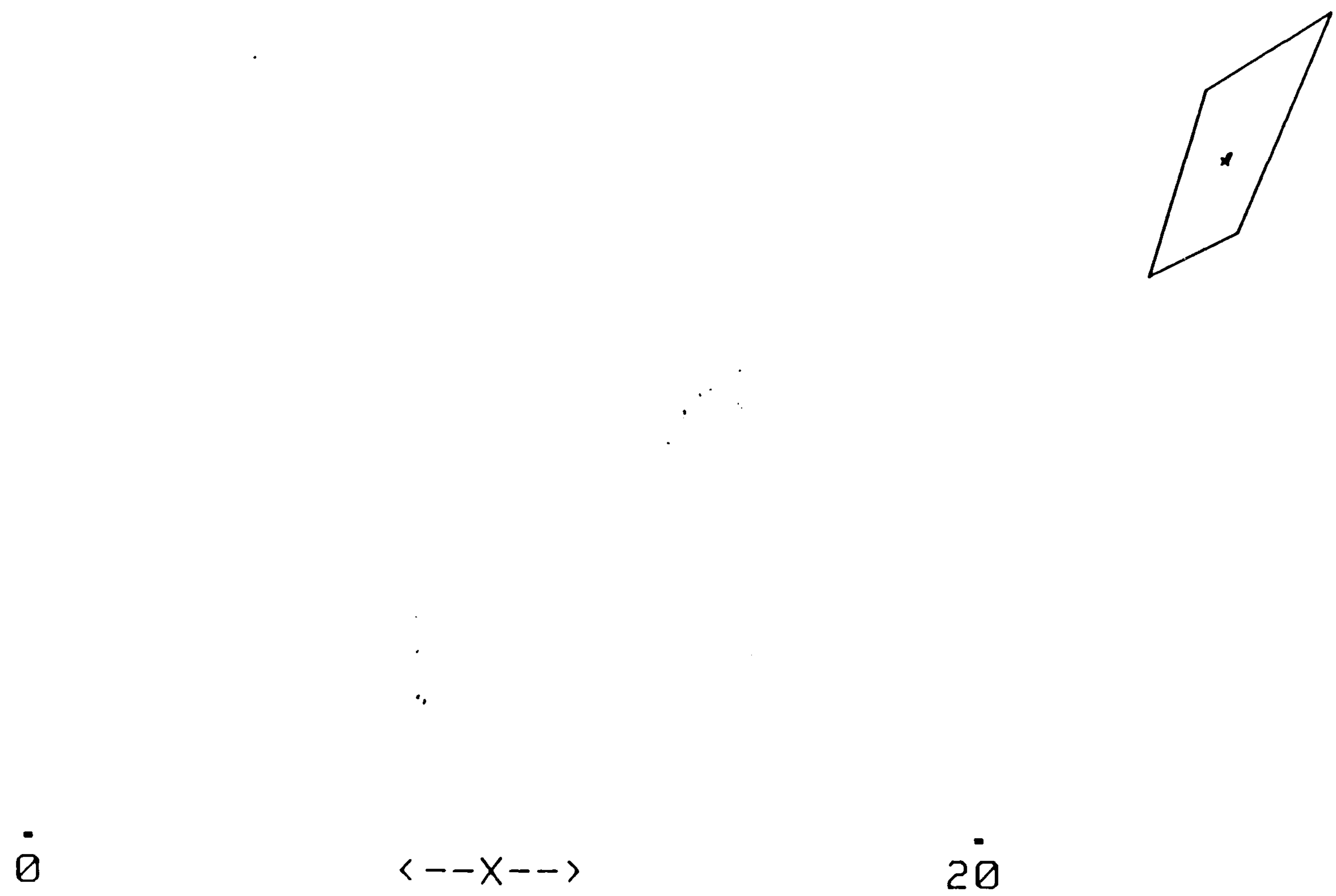


FIGURE 3

2 LOB ANALYSIS

σ=1 degree



σ has shrunk by a factor of three - - the bias region has shrunk by a factor of nine.

FIGURE 4

Emitter is at (28 , 35)

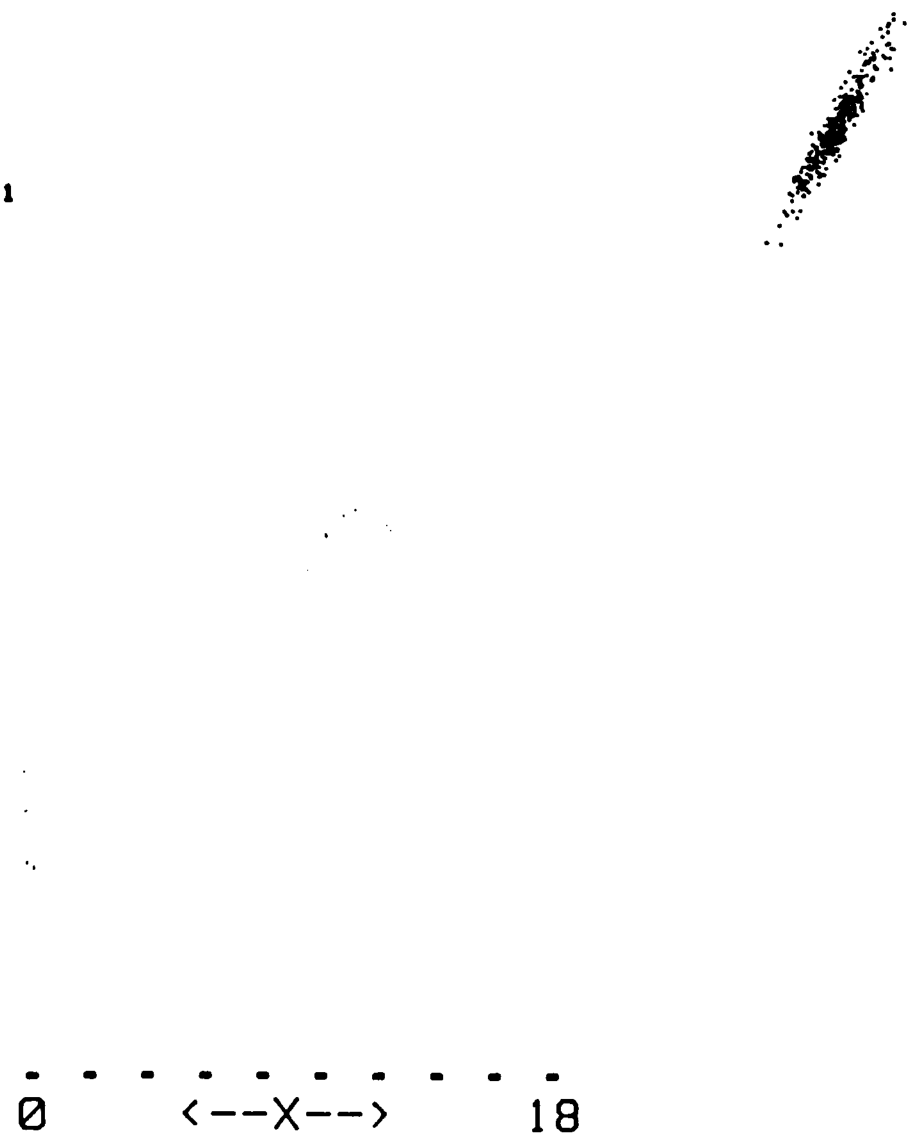
Perpendicular Method

Xbar= 27.754946567

Ybar= 34.5374225811

Angular error,sigma= 1

Magnitude of
  bias vector= .523



EFFECTS OF THIN ANGLES AND SIGMA:

Bias can be large if the angles between the bearings are small or if sigma is large

FIGURE 5

Emitter is at (28 , 35)

Perpendicular Method

Xbar= 25.923139219
Ybar= 31.1072846349

Angular error,sigma= 3
Magnitude of
  bias vector= 4.41

0 <--X--> 18

EFFECTS OF THIN ANGLES AND SIGMA:

If sigma is tripled, then the magnitude of bias will increase by a factor of nine

FIGURE 6

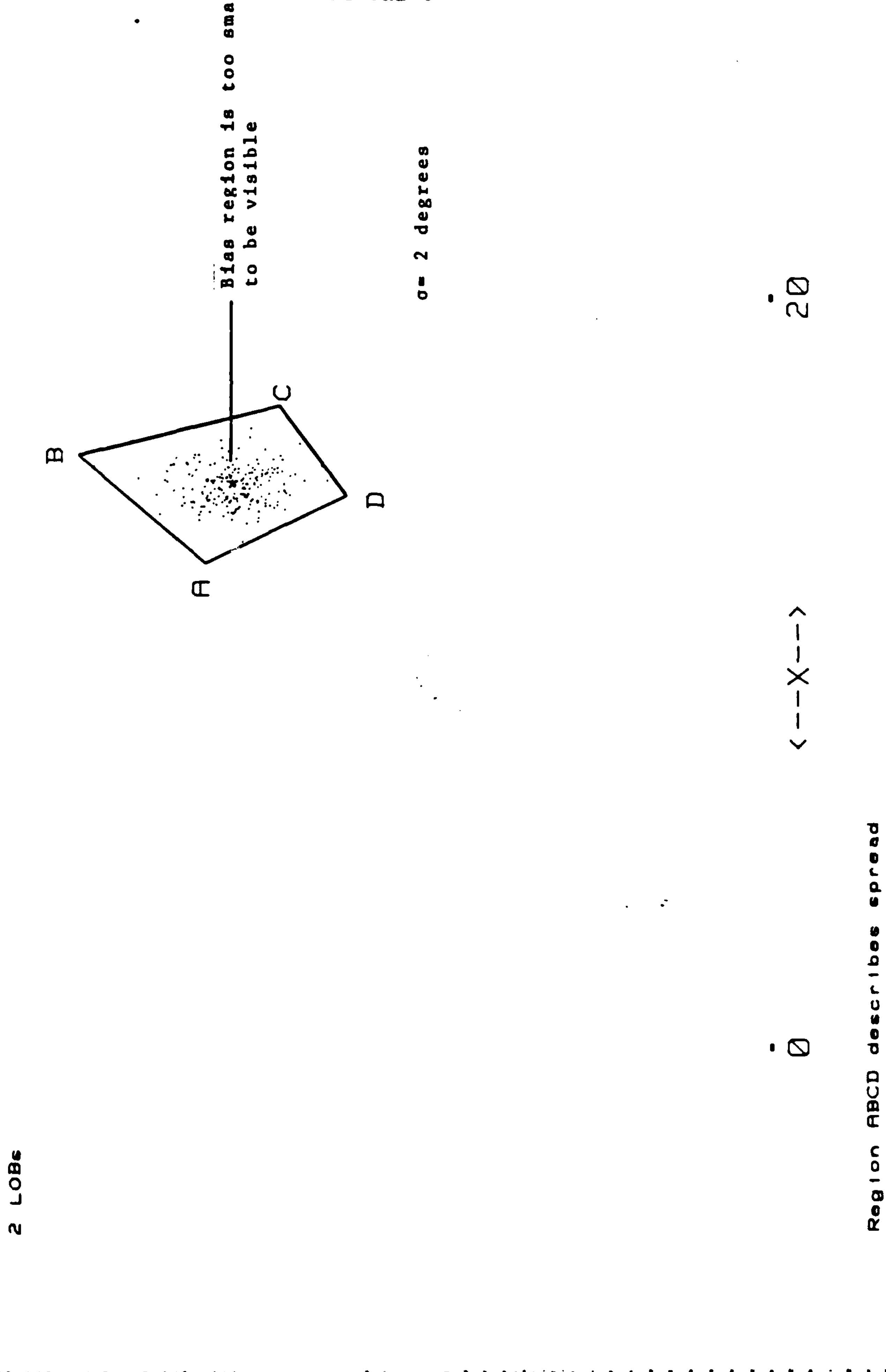
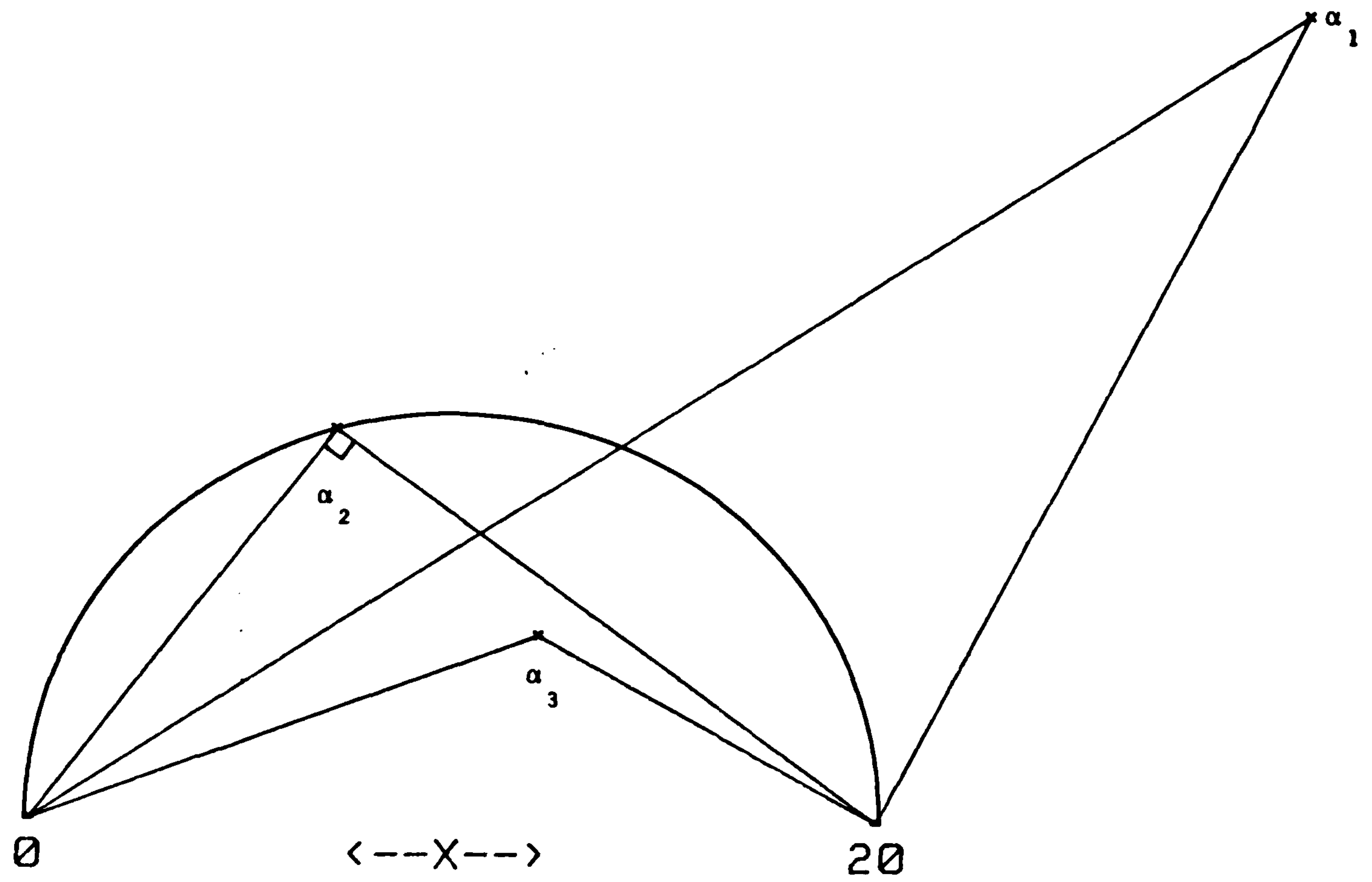

Bias region is too small
to be visible
σ= 2 degrees
A
B
C
D
20
<--X-->
0
2 LOBs
Region ABCD describes spread

FIGURE 7

2 LOB ANALYSIS



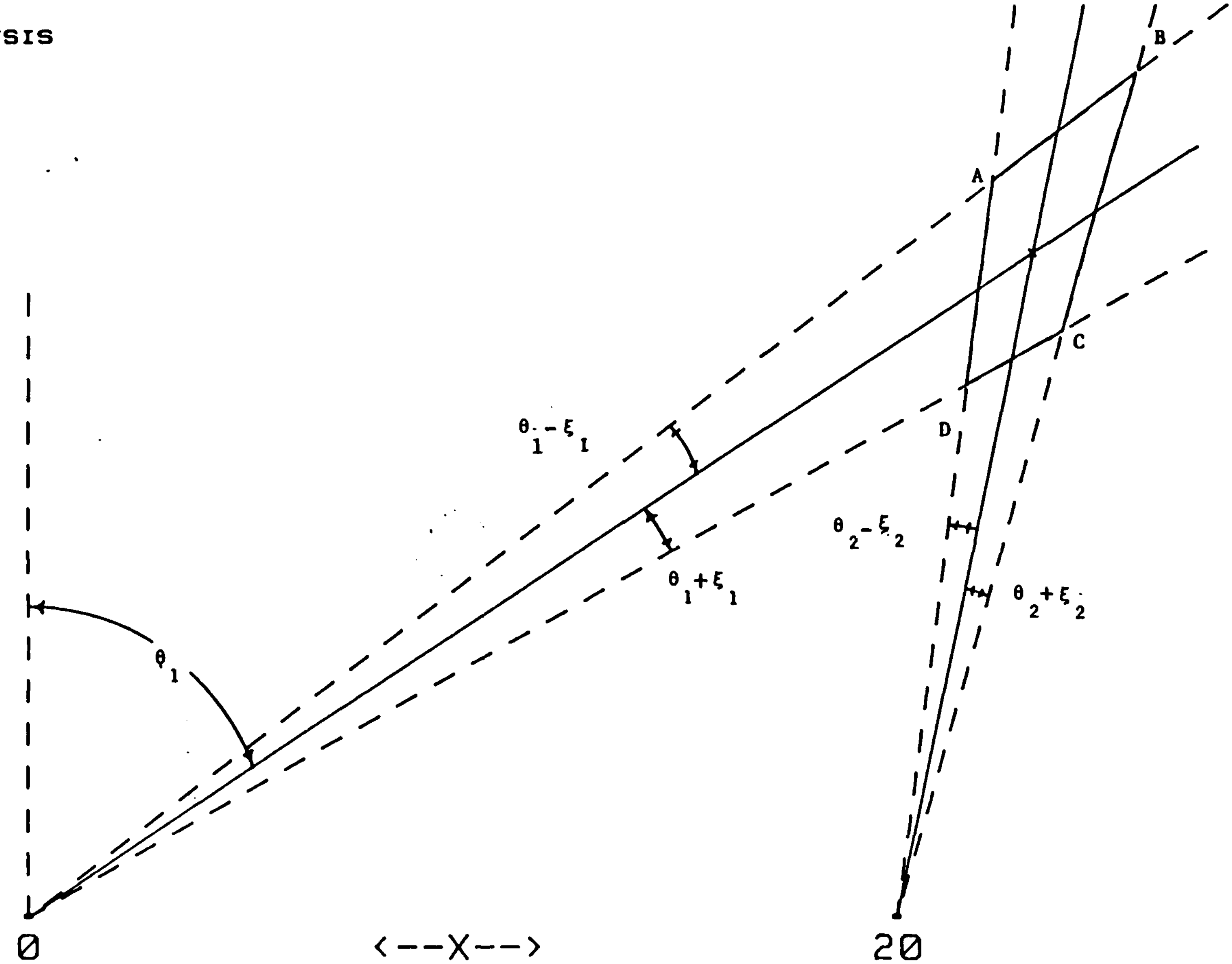
Estimates of emitters outside the circle are biased long (α_1 < 90 degrees)

Estimates of emitters on the circle are unbiased (α_2 = 90 degrees)

Estimates of emitters inside the circle are biased short (α_3 > 90 degrees)

FIGURE 8

2 LOB ANALYSIS



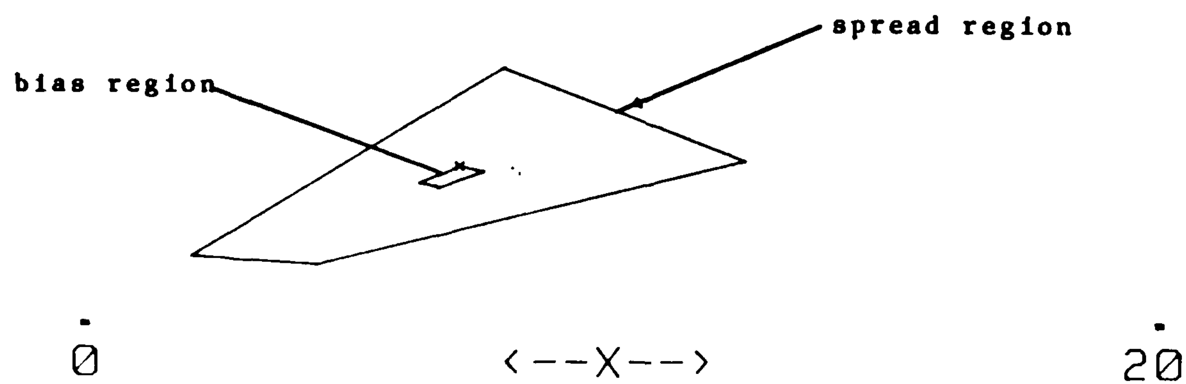
θ_i is measured from true north to the true bearing

ξ_i is the angular error from each sensor. $\xi_i = 3\sigma$

Region ABCD describes the spread of all possible estimates within the given angular error

FIGURE 9

2 LOB ANALYSIS



When the angle between the true bearings is greater than 90 degrees, bias will point short of the emitter

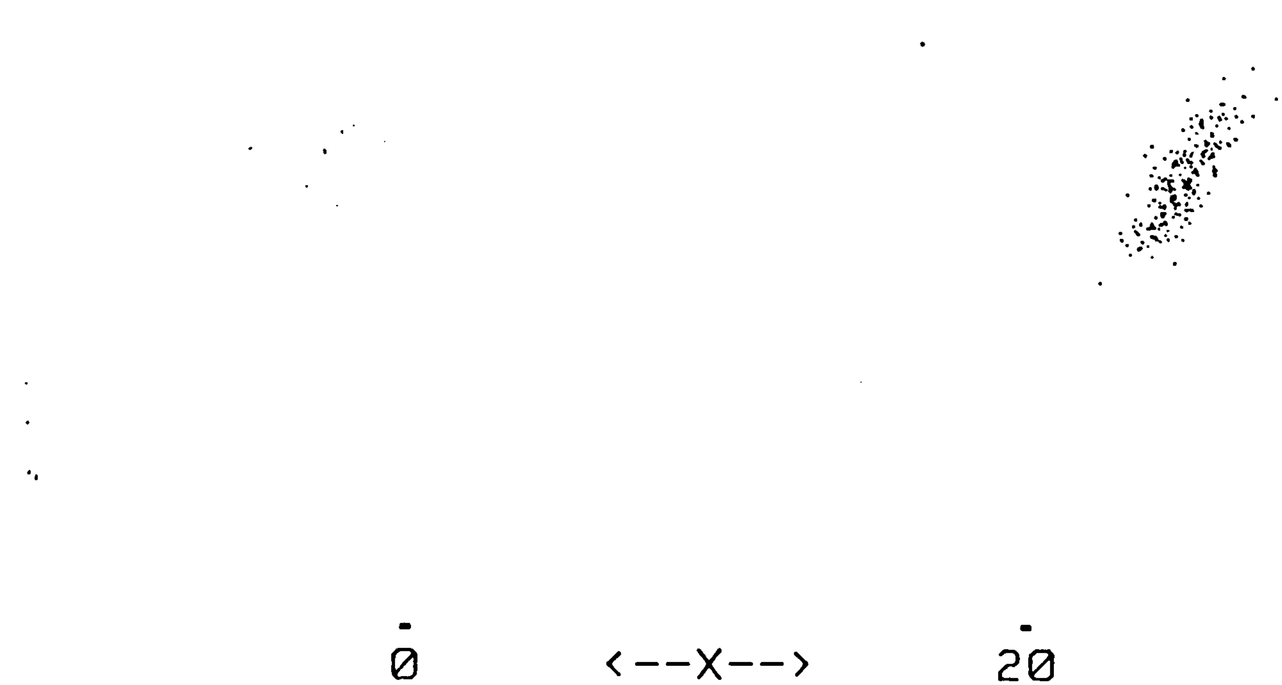
FIGURE 10

Emitter is at (25 , 15)

Min Angular Error

XBAR= 25.0123578442
YBAR= 15.1591916378

2 SENSORS



EFFECTS OF INCREASES IN SAMPLE SIZE:
Comparison of bias using 2 LOBs and 10 LOBs (1 or 5 sensors at each location)
Min Angular Error bias approaches zero as sample size increases

FIGURE 11

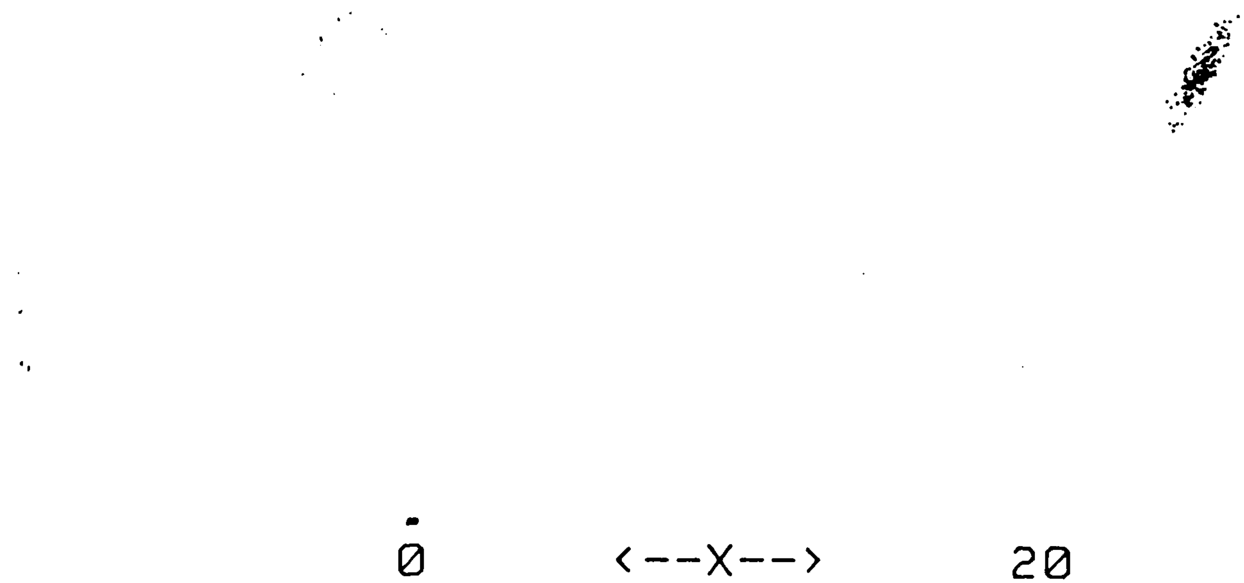
Emitter is at (25 , 15)

Min Angular Error

XBAR= 25.0241294225

YBAR= 15.0510165405

10 SENSORS



EFFECTS OF INCREASES IN SAMPLE SIZE:

Comparison of bias using 2 LOBs and 10 LOBs (1 or 5 sensors at each location)

Min Angular Error bias approaches zero as sample size increases

FIGURE 12

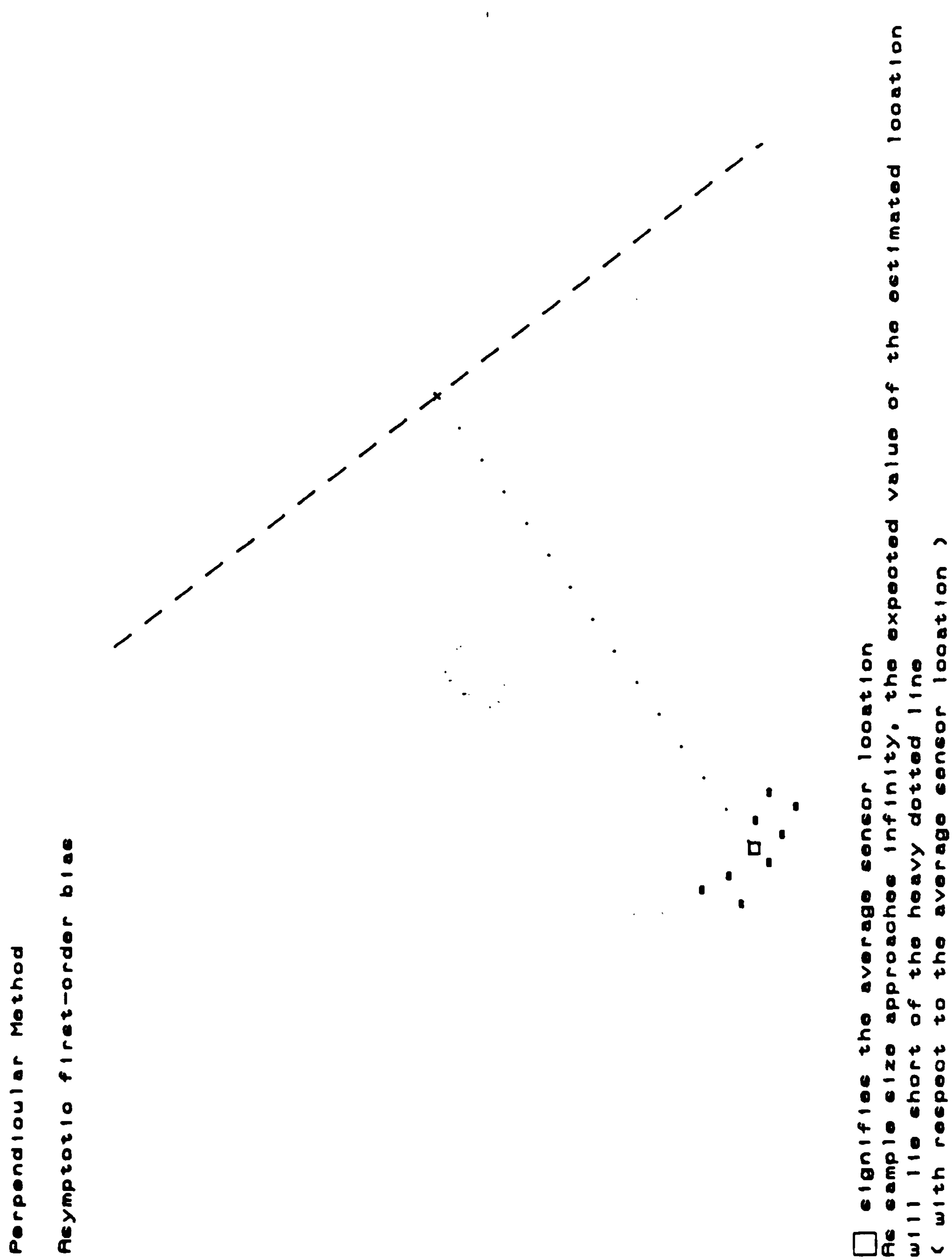
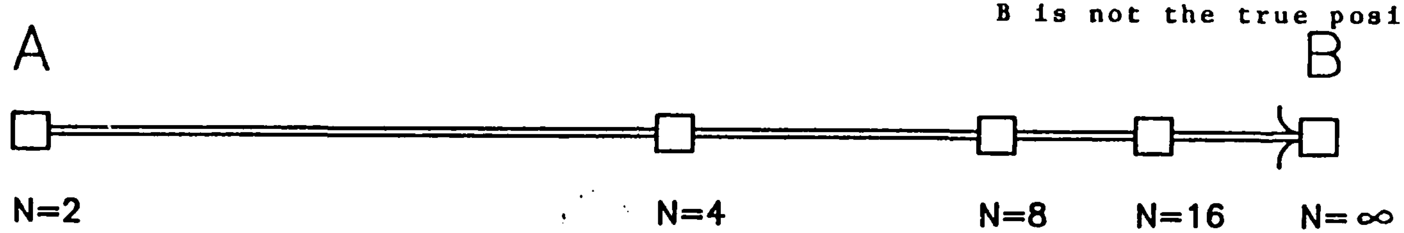

Perpendicular Method
Asymptotic first-order bias
□ signifies the average sensor location
As sample size approaches infinity, the expected value of the estimated location
will lie short of the heavy dotted line
(with respect to the average sensor location)

FIGURE 13

Expected Value for the Estimated Location*

(movement with respect to sample size)



(For the Perpendicular Method B is not the true position)

N= number of LOBs

□ = position (relative to sample size) of the expected value of the estimated location

* This analysis is valid for three of the methods (all but the Weighted Perpendicular). Also, the result assumes that as N increases, there are no significant changes to configuration.

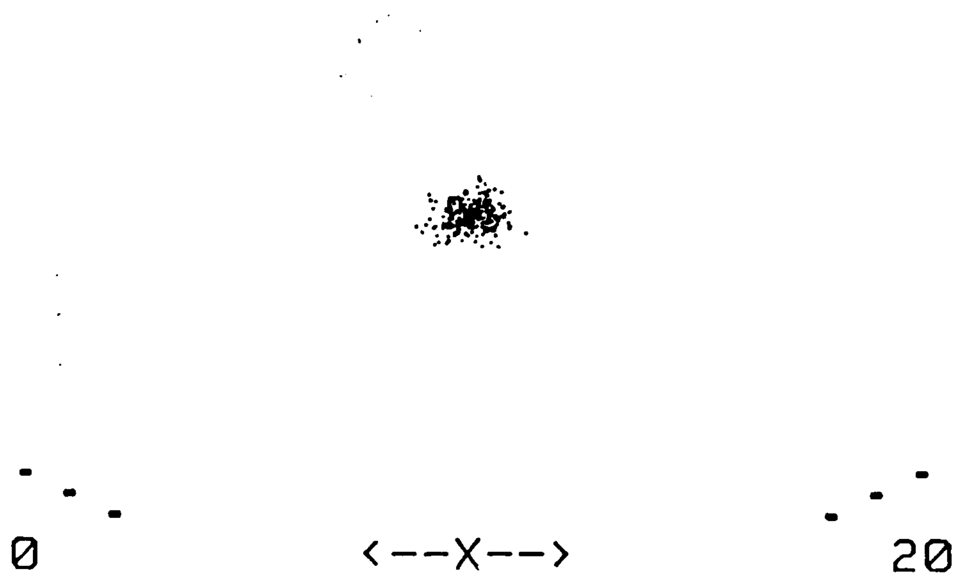
FIGURE 14

Emitter is at (10 , 7)

Min Angular Error

Xbar= 10.0142129925

Ybar= 6.98378394883



SYMMETRIC SCENARIO: MINIMIZATION OF ANGULAR ERROR

If all of the angles between bearings from symmetric sensors are greater than 90 degrees (as shown), bias will point short along the line of symmetry

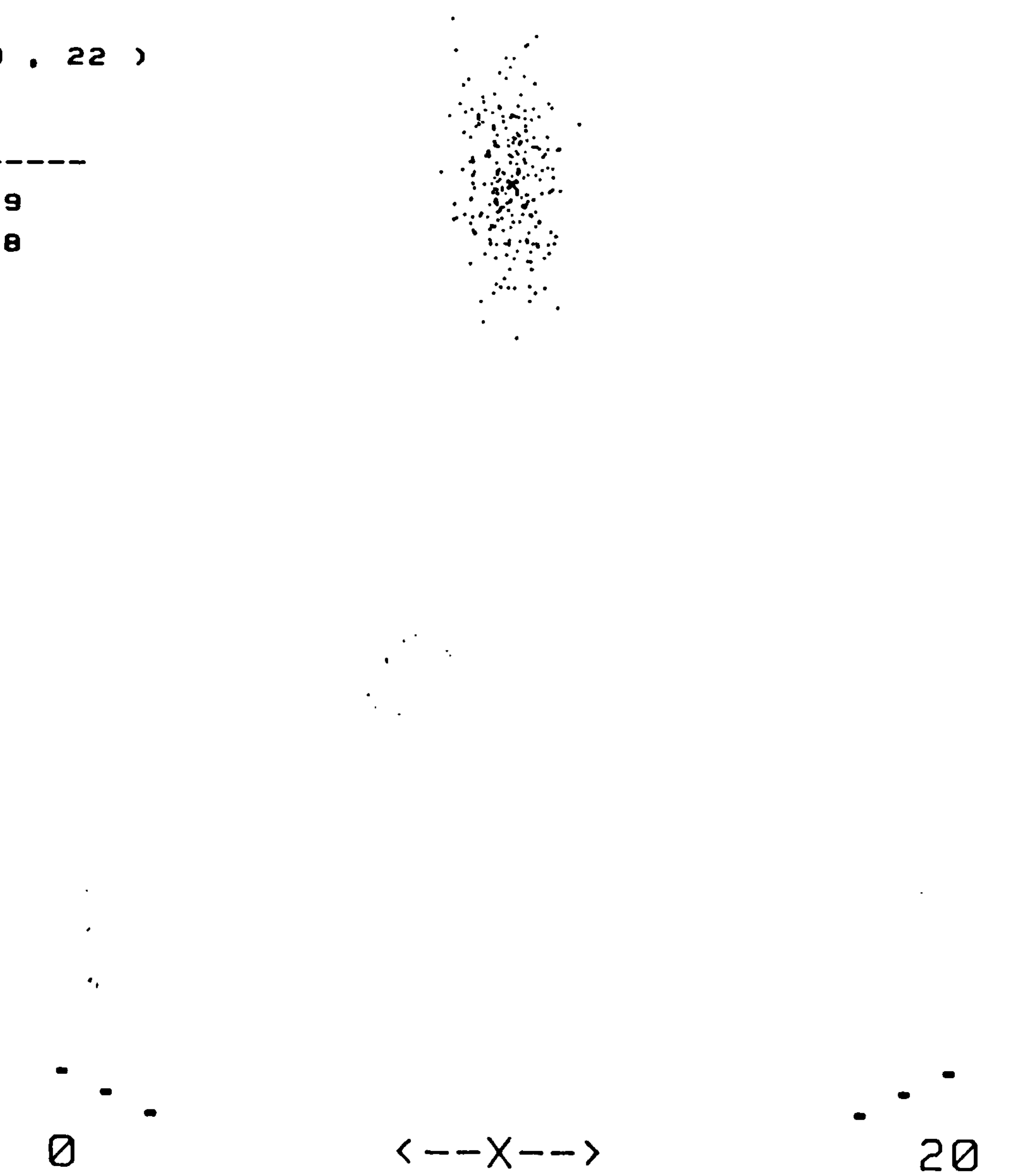
FIGURE 15

Emitter is at (10 , 22)

Min Angular Error

Xbar= 10.0098603689

Ybar= 22.0738873278



SYMMETRIC SCENARIO: MINIMIZATION OF ANGULAR ERROR

If all of the angles between bearings from symmetric sensors are less than 90 degrees (as shown), bias will point long along the line of symmetry (first-order bias =0 when angles equal 90 degrees)

FIGURE 16

Emitter is at (10 , 3.4)

Perpendicular Method

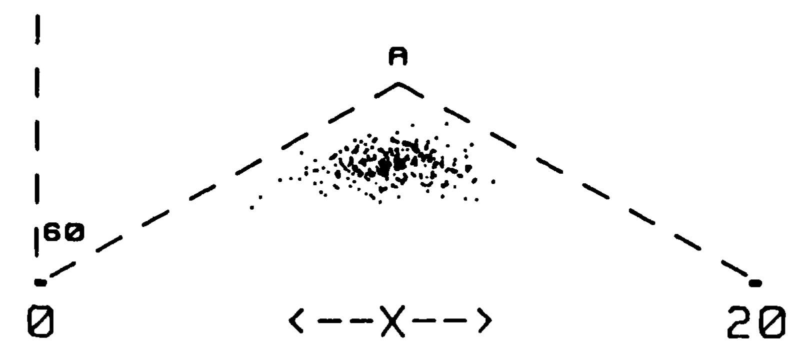
XBAR= 9.98093116779

YBAR= 3.3507548481

2 LOBs

(1 at each location)

Angular error, Sigma= 3



EFFECTS OF INCREASES IN SAMPLE SIZE:

Perpendicular symmetric bias.

If the LOBs form angles > 60 degrees (emitters below Point A), increases in sample size will cause bias to shrink towards the emitter

Note the increase in Ybar (thus less bias) as sample size increases

FIGURE 17

Emitter is at (10 , 3.4)

Perpendicular Method

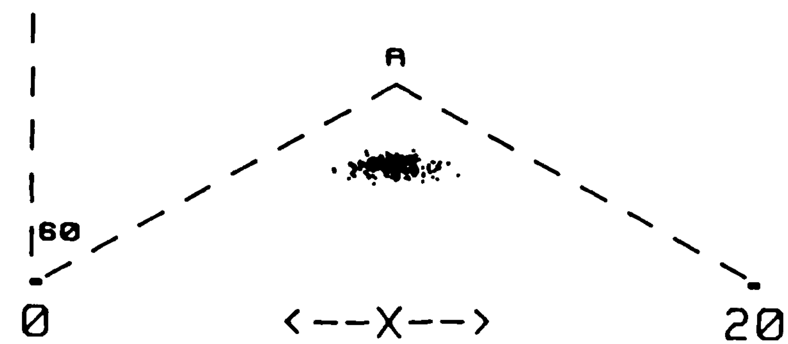
XBAR= 9.99338474716

YBAR= 3.38221498946

10 LOBs

(5 at each location)

Angular error, Sigma= 3



EFFECTS OF INCREASES IN SAMPLE SIZE:

Perpendicular symmetric bias.

If the LOBs form angles > 60 degrees (emitters below Point A), increases in sample size will cause bias to shrink towards the emitter

Note the increase in Ybar (thus less bias) as sample size increases

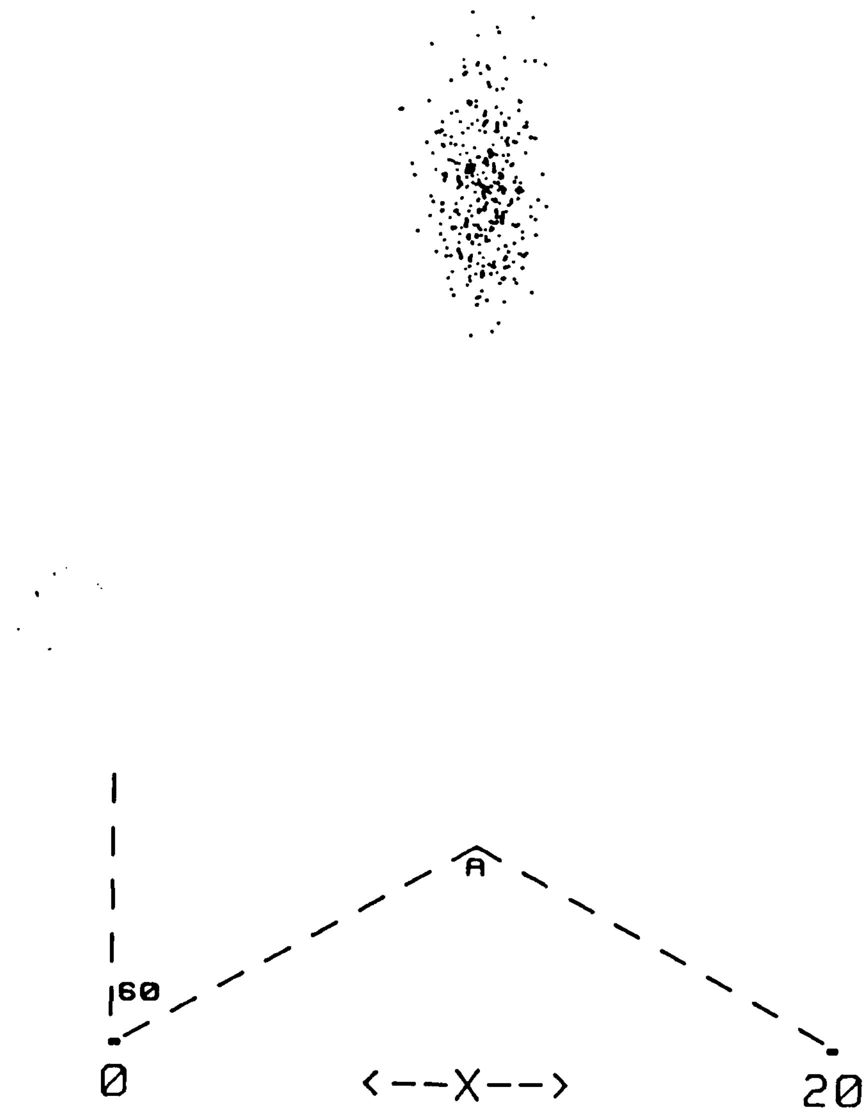
FIGURE 18

Emitter is at (10 , 25)

Perpendicular Method

XBAR= 10.0016336632
YBAR= 25.1142292948

2 LOBs
(1 at each location)
Angular error, Sigma= 2



EFFECTS OF INCREASES IN SAMPLE SIZE:
Perpendicular symmetric bias
If the LOBs form angles < 60 degrees (emitters above Point A), increases in sample size will cause bias to move short from its previous location
Note the decrease in Ybar (and increase in bias) as sample size increases

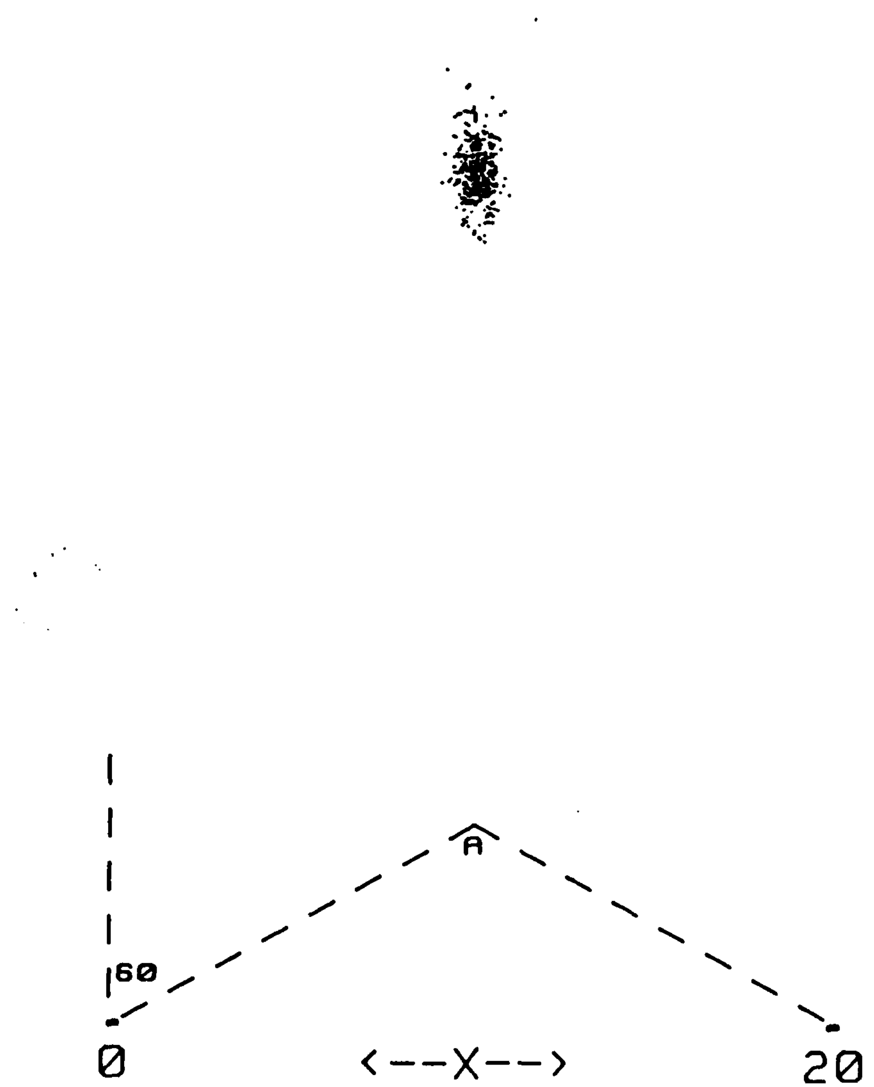
FIGURE 19

Emitter is at (10 , 25)

Perpendicular Method

XBAR= 9.99919994367
YBAR= 24.8342743511

10 LOBs
(5 at each location)
Angular error, Sigma= 2



EFFECTS OF INCREASES IN SAMPLE SIZE:
Perpendicular symmetric bias
If the LOBs form angles < 60 degrees (emitters above Point A), increases in sample size will cause bias to move short from its previous location
Note the decrease in Ybar (and increase in bias) as sample size increases

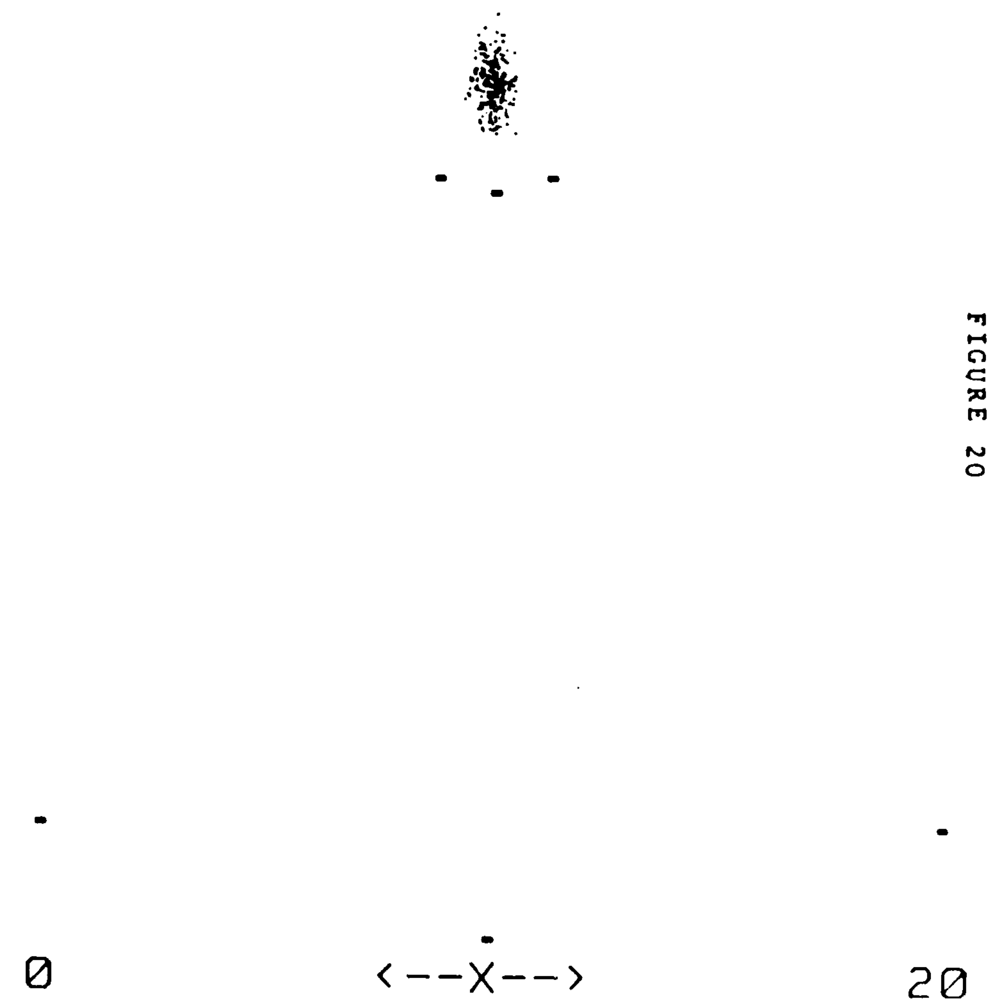
FIGURE 20

Emitter is at (10 , 20)

Perpendicular Method

XBAR= 10.0003717847
YBAR= 19.9518124814

6 SENSORS



DOUBLE ARC SET-UP:
Comparison of Weighted Perpendicular and Perpendicular methods
Perpendicular Method emphasizes the data from the farther arc

FIGURE 21

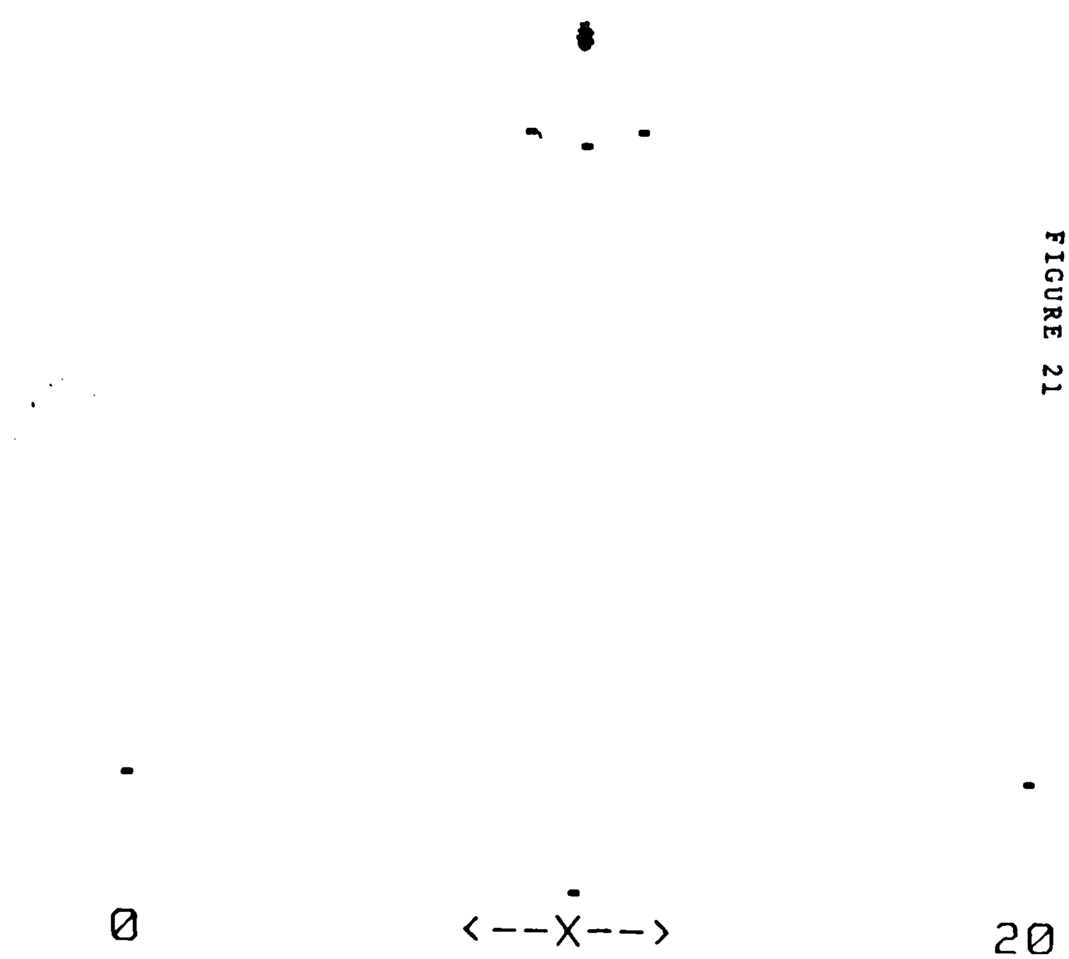
Emitter is at (10 , 20)

Weighted Perpendicular

XBAR= 9.99941806215

YBAR= 19.9966036418

6 SENSORS



DOUBLE ARC SET-UP:
Comparison of Weighted Perpendicular and Perpendicular methods
Weighted Perpendicular emphasizes the data from the closer arc

FIGURE 22

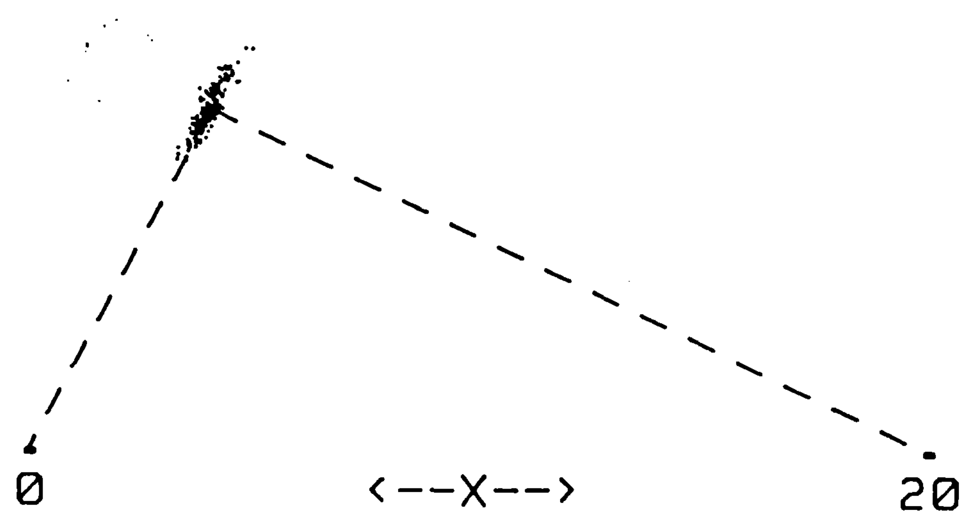
Emitter is at (4 , 8)

Min Angular Error

XBAR= 4.00405619594
YBAR= 8.00970743968

4 LOBs
-- 3 from (0,0)
-- 1 from (20,0)

Angular error, Sigma, varies
from sensor to sensor



90 DEGREE GENERAL UNBIAS: Min Angular Error Method
When the angle between the true bearings is 90 degrees, first-order bias equals 0, regardless of angular error or how many LOBs are taken from each sensor
(The small amount of bias shown in this diagram is due to the fact that an insufficient number of simulations were run in order to provide a clearer picture)

MATHEMATICAL APPENDIX

Formulas for bias have been obtained for the four algorithms discussed in this report. Three of the algorithms have been characterized by least squares while the bias formula for the Weighted Perpendicular method has been derived separately at the end of this section.

2-Dimensional Least Squares Fix Algorithms

Least Squares Algorithms are algorithms that estimate parameters by minimizing a sum of squared terms (usually a measure of error) where each term depends only on the parameters and one observation. For our application the parameters are the x and y coordinates of the location estimate and the observations are LOBs.

Definitions

(X,Y) = true location of the emitter

$\theta_k(X,Y)$ = true bearing from the kth sensor to the emitter (X,Y)

$\hat{\theta}_k$ = observed bearing from the kth sensor (multiple readings are treated as coming from different sensors)

$(x,y) = (x(\hat{\theta}_1,\ldots,\hat{\theta}_n),y(\hat{\theta}_1,\ldots,\hat{\theta}_n))$ = estimated location of the emitter

σ_k = standard deviation of the angular measurement of the kth LOB in radians (multiply by $\pi/180$ if in degrees)

ϵ_k = error in kth LOB $(\hat{\theta}_k-\theta_k(X,Y))$

(x_k,y_k) = sensor location of kth LOB

$r_k^2 = (x-x_k)^2+(y-y_k)^2 \qquad R_k^2=(X-x_k)^2+(Y-y_k)^2$

LIKELIHOOD MINIMIZATION TYPE FIX METHODS

Part of the analysis of these methods is independent of which method is being analyzed.

Extra definitions that apply to Likelihood based methods

$L_k = L_k(x,y,\hat{\theta}_k)$ = the 'squared error term' corresponding to the kth LOB and a location estimate of (x,y) for the least square method.

$L(x,y,\hat{\theta}_1,\dots,\hat{\theta}_n) = \sum_{i=1}^{n} L_k$ = the sum of squares that the method minimizes

Assume that x,y are defined implicitly as the minimum of

$L=\Sigma\, L_k(x,y,\hat{\theta}_k)$ (for summation notation, see * below)

in the sense that $\partial L/\partial x=\partial L/\partial y=0$

The evaluated partial derivatives of L_k will be denoted as follows (recall that evaluated partial derivatives are used in a Taylor Series).

Let $b_k=\partial L_k/\partial x$ $c_k=\partial L_k/\partial y$ $d_k=\partial^2 L_k/\partial x^2$ $e_k=\partial^2 L_k/\partial x\partial y$ $f_k=\partial^2 L_k/\partial y^2$

$g_k=\partial^3 L_k/\partial x^3$ $h_k=\partial^3 L_k/\partial x^2\, y$ $i_k=\partial^3 L_k/\partial x\partial y^2$ $j_k=\partial^3 L_k/\partial y^3$

Let $'$ notation represent derivatives with respect to θ_k for b_k to f_k.

Then, since $\partial L/\partial x=0$, $\Sigma b_k=0$ and since $\partial L/\partial y=0$, $\Sigma c_k=0$

$$0=\partial/\partial\varepsilon_i(\partial L/\partial x)=\Sigma\partial/\partial\varepsilon_i\cdot b_k=\Sigma(\partial x/\partial\varepsilon_i\cdot\partial b_k/\partial x+\partial y/\partial\varepsilon_i\cdot\partial b_k/\partial y+\partial\theta_k/\partial\varepsilon_i\cdot\partial b_k/\partial\theta_k)$$
$$=\partial x/\partial\varepsilon_i(\Sigma d_k)+\partial y/\partial\varepsilon_i(\Sigma e_k)+b_i'$$

$$0=\partial^2/\partial\varepsilon_i^2(\partial L/\partial x)=\partial^2 x/\partial\varepsilon^2(\Sigma d_k)+(\partial x/\partial\varepsilon_i)^2(\Sigma\partial d_k/\partial x)+(\partial x/\partial\varepsilon_i)(\partial y/\partial\varepsilon_i)(\Sigma\partial d_k/\partial y)$$
$$+\partial x/\partial\varepsilon_i\cdot d_i'+\partial^2 y/\partial\varepsilon_i^2(\Sigma e_k)+(\partial x/\partial\varepsilon_i)(\partial y/\partial\varepsilon_i)(\Sigma\partial e_k/\partial x)+(\partial y/\partial\varepsilon_i)^2(\Sigma\partial e_k/\partial y)$$
$$+\partial y/\partial\varepsilon_i\cdot e_i'+\partial x/\partial\varepsilon_i\cdot\partial b_i'/\partial x+\partial y/\partial\varepsilon_i(\partial b_i'/\partial y)+b_i''$$
$$=\partial^2 x/\partial\varepsilon_i^2(\Sigma d_k)+(\partial x/\partial\varepsilon_i)^2(\Sigma g_k)+(\partial x/\partial\varepsilon_i)(\partial y/\partial\varepsilon_i)(\Sigma h_k)+\partial x/\partial\varepsilon_i\cdot d_i'+\partial^2 y/\partial\varepsilon_i^2(\Sigma e_k)$$
$$+(\partial x/\partial\varepsilon_i)(\partial y/\partial\varepsilon_i)(\Sigma h_k)+(\partial y/\partial\varepsilon_i)^2(\Sigma i_k)+\partial y/\partial\varepsilon_i\cdot e_i'+\partial x/\partial\varepsilon_i\cdot\partial b_i'/\partial x$$
$$+\partial y/\partial\varepsilon_i(\partial b_i'/\partial y)+b_i''$$

* Un-labeled sums are assumed to be $\sum_{k=1}^{n}$ (n equals the number of LOBs).
Sums labeled Σ_i are assumed to be $\sum_{i=1}^{n}$

$$=\partial^2x/\partial\varepsilon_i^2(\Sigma d_k)+\partial^2y/\partial\varepsilon_i^2(\Sigma e_k)+(\partial x/\partial\varepsilon_i)^2(\Sigma g_k)+2(\partial x/\partial\varepsilon_i)(\partial y/\partial\varepsilon_i)(\Sigma h_k)$$
$$+(\partial y/\partial\varepsilon_i)^2(\Sigma i_k)+2(\partial y/\partial\varepsilon_i)e_i'+2(\partial x/\partial\varepsilon_i)d_i'+b_i''$$

Similar argument may be applied to $\partial L/\partial y$ yielding

$$\begin{bmatrix}0\\0\end{bmatrix}=\partial/\partial\varepsilon_i\begin{bmatrix}\partial L/\partial x\\\partial L/\partial y\end{bmatrix}=\begin{bmatrix}\Sigma d_k & \Sigma e_k\\\Sigma e_k & \Sigma f_k\end{bmatrix}\begin{bmatrix}\partial x/\partial\varepsilon_i\\\partial y/\partial\varepsilon_i\end{bmatrix}+\begin{bmatrix}b_i'\\c_i'\end{bmatrix}$$

so, $$\begin{bmatrix}\partial x/\partial\varepsilon_i\\\partial y/\partial\varepsilon_i\end{bmatrix}=\frac{-1}{(\Sigma d_k)(\Sigma f_k)-(\Sigma e_k)^2}\begin{bmatrix}\Sigma f_k & -\Sigma e_k\\-\Sigma e_k & \Sigma d_k\end{bmatrix}\begin{bmatrix}b_i'\\c_i'\end{bmatrix}$$

and

$$\begin{bmatrix}0\\0\end{bmatrix}=\partial^2/\partial\varepsilon_i^2\begin{bmatrix}\partial L/\partial x\\\partial L/\partial y\end{bmatrix}=\begin{bmatrix}\Sigma d_k & \Sigma e_k\\\Sigma e_k & \Sigma f_k\end{bmatrix}\begin{bmatrix}\partial^2x/\partial\varepsilon_i^2\\\partial^2y/\partial\varepsilon_i^2\end{bmatrix}+\begin{bmatrix}\text{TERM 1}\\\text{TERM 2}\end{bmatrix}$$

where

$$\begin{bmatrix}\text{TERM 1}\\\text{TERM 2}\end{bmatrix}=\begin{bmatrix}(\partial x/\partial\varepsilon_i)^2(\Sigma g_k)+2(\partial x/\partial\varepsilon_i)(\partial y/\partial\varepsilon_i)(\Sigma h_k)+(\partial y/\partial\varepsilon_i)^2(\Sigma i_k)\\(\partial x/\partial\varepsilon_i)^2(\Sigma h_k)+2(\partial x/\partial\varepsilon_i)(\partial y/\partial\varepsilon_i)(\Sigma i_k)+(\partial y/\partial\varepsilon_i)^2(\Sigma j_k)\end{bmatrix}$$
$$+\begin{bmatrix}2(\partial y/\partial\varepsilon_i)e_i'+2(\partial x/\partial\varepsilon_i)d_i'+b_i''\\2(\partial y/\partial\varepsilon_i)f_i'+2(\partial x/\partial\varepsilon_i)e_i'+c_i''\end{bmatrix}$$

Thus,

$$\begin{bmatrix}\partial^2x/\partial\varepsilon_i^2\\\partial^2y/\partial\varepsilon_i^2\end{bmatrix}=\frac{-1}{(\Sigma d_k)(\Sigma f_k)-(\Sigma e_k)^2}\begin{bmatrix}\Sigma f_k & -\Sigma e_k\\-\Sigma e_k & \Sigma d_k\end{bmatrix}\begin{bmatrix}\text{TERM 1}\\\text{TERM 2}\end{bmatrix}$$

$$\text{First-order bias}=1/2\cdot\Sigma_i\begin{bmatrix}\partial^2x/\partial\varepsilon_i^2\\\partial^2y/\partial\varepsilon_i^2\end{bmatrix}\cdot\sigma_i^2$$

Particular algorithms use a different L_k from which to derive the partial derivatives used in the formula for bias.

<u>Perpendicular Method</u>

$$L_k=[(x-x_k)^2\cos^2\theta_k+(y-y_k)^2\sin^2\theta_k-2(y-y_k)(x-x_k)\sin\theta_k\cos\theta_k]$$

The evaluated partial derivatives are as follows (evaluated at $\hat{\theta}_k = \theta_k$).

$b_i' = -2(Y-y_i)$

$c_i' = 2(X-x_i)$

$d_i' = -4(Y-y_i)(X-x_i)/R_i^2$

$e_i' = 2((X-x_i)^2-(Y-y_i)^2)/R_i^2$

$f_i' = 4(Y-y_i)(X-x_i)/R_i^2$

$b_i'' = 4(X-x_i)$

$c_i'' = 4(Y-y_i)$

$d_k = 2(Y-y_k)^2/R_k^2$

$e_k = -2(Y-y_k)(X-x_k)/R_k^2$

$f_k = 2(X-x_k)^2/R_k^2$

$g_k = h_k = i_k = j_k = 0$

Minimization of Angular Error Method

$L_k = [\text{Arctan}((x-x_k)/(y-y_k))-\theta_k]^2$

The evaluated partial derivatives are as follows (evaluated at $\hat{\theta}_k = \theta_k$).

$b_i' = -2(Y-y_i)/R_i^2$

$c_i' = 2(X-x_i)/R_i^2$

$d_i' = 4(Y-y_i)(X-x_i)/R_i^4$

$e_i' = 2((Y-y_i)^2-(X-x_i)^2)/R_i^4$

$f_i' = -4(Y-y_i)(X-x_i)/R_i^4$

$b_i'' = 0$

$c_i'' = 0$

$d_k = 2(Y-y_k)^2/R_k^4$

$e_k = -2(Y-y_k)(X-x_k)/R_k^4$

$f_k = 2(X-x_k)^2/R_k^4$

$g_k = -12(X-x_k)(Y-y_k)^2/R_k^6$

$h_k = (8(X-x_k)^2(Y-y_k)-4(Y-y_k)^3)/R_k^6$

$i_k = (8(X-x_k)(Y-y_k)^2-4(X-x_k)^3)/R_k^6$

$j_k = -12(X-x_k)^2(Y-y_k)/R_k^6$

Sine of Error Minimization Method

The partial derivatives for this method are identical to the ones just shown for the Minimization of Angular Error method. Thus, first-order bias is identical for both methods. However, the L_k is as follows.

$$L_k = [(x-x_k)^2\cos^2\theta_k+(y-y_k)^2\sin^2\theta_k-2(y-y_k)(x-x_k)\sin\theta_k\cos\theta_k]/[(x-x_k)^2+(y-y_k)^2]$$

Notice the similarity of this term with the L_k for the Perpendicular Method. It is the same except that this one is divided by $(x-x_k)^2+(y-y_k)^2$, which is simply r_k^2, the distance squared from the kth sensor to the estimated location of the emitter.

Infinite Number of LOBs

First-order bias as the number of LOBs approaches infinity is easily characterized. Each of the terms in the general formula has a sum in the numerator and in the denominator, except for the last terms in Term 1 and Term 2: b_i'' and c_i''. As N goes to infinity, those terms with sums in both numerator and denominator will go to zero and the asymptotic first-order bias will be:

$$1/2\cdot \Sigma_i \frac{-1}{(\Sigma d_k)(\Sigma f_k)-(\Sigma e_k)^2} \begin{bmatrix} \Sigma f_k & -\Sigma e_k \\ -\Sigma e_k & \Sigma d_k \end{bmatrix} \begin{bmatrix} b_i'' \\ c_i'' \end{bmatrix} \cdot \sigma_i^2$$

For the two methods Minimization of Angular Error and the Sine Of Error Minimization, the b_i'' and the c_i'' terms are zero so the two methods are asymptotically unbiased for first-order.

The Perpendicular method, however, is asymptotically biased. When the proper terms are substituted into the formula, the result is as follows.

With a change of coordinates, the y-axis can be made to go through the emitter and the average sensor location, and b_i'' can be made to equal zero (without loss of generality). Then, the x-component of bias will be ignored, and only the y-component examined. The y-component of asymptotic first-order bias will be as follows:

$$1/2\cdot \Sigma_i \frac{-1}{(\Sigma d_k)(\Sigma f_k)-(\Sigma e_k)^2} \quad \Sigma d_k \cdot c_i'' \cdot \sigma_i^2$$

The sign of this term will always be negative so first-order bias will point towards the average sensor location.

A similar analysis may be performed in order to determine asymptotic first-order bias for the Weighted Perpendicular Method. Like the Perpendicular Method, the Weighted Perpendicular Method produces biased estimates as sample size approaches infinity. In general, this first-order asymptotic bias will point short of the true location (details are discussed in the Weighted Perpendicular section).

General Symmetric Case

Symmetry of sensors produces the following.

$\Sigma e_k = \Sigma g_k = \Sigma i_k = 0$

Then, $\partial x/\partial \varepsilon_i = -b_i'/\Sigma d_k$ and $\partial y/\partial \varepsilon_i = -c_i'/\Sigma f_k$

and,

$$\begin{bmatrix} \partial^2 x/\partial \varepsilon_i^2 \\ \partial^2 y/\partial \varepsilon_i^2 \end{bmatrix} = \begin{bmatrix} 1/\Sigma d_k & 0 \\ 0 & 1/\Sigma f_k \end{bmatrix} \begin{bmatrix} 2(\partial x/\partial \varepsilon_i)(\partial y/\partial \varepsilon_i)(\Sigma h_k)+2(\partial y/\partial \varepsilon_i)e_i'+2(\partial x/\partial \varepsilon_i)d_i'+b_i'' \\ (\partial x/\partial \varepsilon_i)^2(\Sigma h_k)+(\partial y/\partial \varepsilon_i)^2(\Sigma j_k)+2(\partial y/\partial \varepsilon_i)f_i'+2(\partial x/\partial \varepsilon_i)e_i'+c_i'' \end{bmatrix}$$

If it is also assumed that the opposite pairs of σ_is are the same, then all of the terms in the top component of the second matrix will be zero after they are summed on i. Then, the x-component of bias will be zero and the y-component will be as follows.

Symmetric bias (y-component)= $1/2 \cdot \Sigma_i(\partial^2 y/\partial \varepsilon_i^2) \cdot \sigma_i^2$ where

$$\partial^2 y/\partial \varepsilon_i^2 = (-b_i/\Sigma d_k)^2(\Sigma h_k)+(-c_i/\Sigma f_k)^2(\Sigma j_k)+2(-c_i/\Sigma f_k)f_i'+2(-b_i/\Sigma d_k)e_i'+c_i''$$

If less assumptions are made (for instance, if the opposite pairs of σ_is are not the same), then intermediate results may also be derived which are useful.

Perpendicular Method

It will be assumed here and for the rest of the symmetric cases that opposite pairs of σ_is are the same. Then, the x-component of first-order bias is zero and the y-component of first-order bias is as follows.

$$\Sigma_i[\partial^2 y/\partial \varepsilon_i^2]\cdot\sigma_i^2 = \Sigma_i[-\sigma_i^2/\Sigma f_k(-2\cdot b_i'\cdot e_i'/\Sigma d_k -2\cdot f_i'\cdot c_i'/\Sigma f_k +c_i'')]$$

With more than 2 LOBs, bias is always negative. However, as sample size increases, the direction that bias will move in is dependent upon the size of the angles between the true lines of bearings. Bias will become less negative if the angles between bearings are all greater than 120 degrees and bias will become more negative if the angles are less than 120 degrees. If all of the angles are equal to 120 degrees, then bias will not move from its 2 LOB position to its infinite LOB position. The proof of this involves determining the sign of the bias term when the angles between bearings are either less than or greater than 120 degrees.

When the absolute value of all of the angles of the true bearings from each sensor are greater than or equal to 120 degrees,

$b_i'e_i' = -4r_i(\cos\theta_i - \cos^3\theta_i)$ $c_i'f_i' = 8r_i(\cos\theta_i - \cos^3\theta_i)$ $\cos\theta_i \leq 1/2$

$d_k \leq 1/2$ $f_k \geq 3/2$

$$\Sigma d_k \le N/2 \le 1/3 \cdot 3N/2 \le 1/3 \cdot \Sigma f_k$$

$$\Sigma f_k \ge 3\Sigma d_k \quad \text{so} \quad 1/\Sigma f_k \le 1/(3\cdot\Sigma d_k) \quad \text{and} \quad -1/\Sigma f_k \ge -1/(3\cdot\Sigma d_k)$$

$$-b_i'e_i'/\Sigma d_k - c_i'f_i'/\Sigma f_k \ge -3b_i'e_i'/(3\cdot\Sigma d_k) - c_i'f_i'/(3\cdot\Sigma d_k)$$

the right side of inequality $=(4\cos\theta_i - 16\cos^3\theta_i)/(3\cdot\Sigma d_k) \ge 0$

Bias has one more negative sign in it, therefore the above term is less than or equal to zero.

In order to find out the movement of bias as sample size increases, for the opposite case when all of the angles are less than or equal to 120 degrees, the same argument is applied except that

$$\cos\theta_i \ge 1/2 \quad d_k \ge 1/2 \quad f_k \le 3/2 \quad \text{and} \quad -1/\Sigma f_k \ge -1/(3\cdot\Sigma d_k)$$

so that bias is found to be greater than or equal to zero. Combining these two results provides for the conclusion of its behavior at 120 degrees; the movement is zero.

Minimization of Angular Error Method

After insertion of the proper terms, first-order symmetric bias for sensors with the same standard deviation will be as follows:

$$\frac{2\sigma^2}{\Sigma(X-x_k)^2/R_k^4} \quad \frac{\Sigma\Sigma\ (X-x_k)^2(Y-y_k)[(Y-y_j)^2-(X-x_j)^2]/(R_k^6\cdot R_j^4)}{\Sigma\Sigma\ (Y-y_k)^4(X-x_j)^2/(R_k^4\cdot R_j^4)}$$

The sign of this expression depends on if the angles between bearings are less than or greater than 90 degrees. At 90 degrees, first-order symmetric bias is zero.

Weighted Perpendicular (Not Definable in Terms of Least Squares)

The Weighted Perpendicular Method is not definable as a least square's method and so a separate formula needs to be derived for first-order bias. It is necessary to establish notation for this case.

Definitions

$$a_i = \cos^2\hat\theta_i \quad A_i = \cos^2\theta_i(X,Y)=(Y-y_i)^2/R_i^2 \quad a = \Sigma a_i/r_i^2 \quad A = \Sigma A_i/R_i^2$$

$$b_i = \sin^2\hat\theta_i \quad B_i = \sin^2\theta_i(X,Y)=(X-x_i)^2/R_i^2 \quad b = \Sigma b_i/r_i^2 \quad B = \Sigma B_i/R_i^2$$

$$c_i = \sin\hat\theta_i\cos\hat\theta_i \quad C_i=(X-x_i)(Y-y_i)/R_i^2 \quad c = \Sigma c_i/r_i^2 \quad C = \Sigma C_i/R_i^2$$

$$d_i = a_i x_i - c_i y_i \quad D_i = A_i x_i - C_i y_i \quad d = \Sigma d_i/r_i^2 \quad D = \Sigma D_i/R_i^2$$

$$e_i = -c_i x_i + b_i y_i \quad E_i = -C_i x_i + B_i y_i \quad e = \Sigma e_i/r_i^2 \quad E = \Sigma E_i/R_i^2$$

$$s_i = \begin{bmatrix} a_i & -c_i \\ -c_i & b_i \end{bmatrix} \qquad \Sigma s_i/r_i^2 = \begin{bmatrix} a & -c \\ -c & b \end{bmatrix}$$

$$(x,y) = (\Sigma s_i/r_i^2)^{-1}(\Sigma s_i/r_i^2 \begin{bmatrix} x_i \\ y_i \end{bmatrix}) = \frac{1}{ab-c^2} \begin{bmatrix} b & c \\ c & a \end{bmatrix} \begin{bmatrix} d \\ e \end{bmatrix}$$

First Partials:

$$\partial r_i^2/\partial\hat{\theta}_k = 2(x-x_i)(\partial x/\partial\hat{\theta}_k) + 2(y-y_i)(\partial y/\partial\hat{\theta}_k)$$

$$\partial a/\partial\hat{\theta}_k = -2c_k/r_k^2 - \Sigma(a_i/r_i^4 \cdot \partial r_i^2/\partial\hat{\theta}_k)$$

$$\partial b/\partial\hat{\theta}_k = 2c_k/r_k^2 - \Sigma(b_i/r_i^4 \cdot \partial r_i^2/\partial\hat{\theta}_k)$$

$$\partial c/\partial\hat{\theta}_k = (a_k-b_k)/r_k^2 - \Sigma(c_i/r_i^4 \cdot \partial r_i^2/\partial\hat{\theta}_k)$$

$$\partial d/\partial\hat{\theta}_k = [-2c_kx_k + (b_k-a_k)y_k]/r_k^2 - \Sigma(d_i/r_i^4 \cdot \partial r_i^2/\partial\hat{\theta}_k)$$

$$\partial e/\partial\hat{\theta}_k = [(b_k-a_k)x_k + 2c_ky_k]/r_k^2 - \Sigma(d_i/r_i^4 \cdot \partial r_i^2/\partial\hat{\theta}_k)$$

Differentiating the defining equation:

$$\begin{bmatrix} \partial x/\partial\hat{\theta}_k \\ \partial y/\partial\hat{\theta}_k \end{bmatrix} = \frac{-1}{(ab-c^2)^2}(a(\partial b/\partial\theta_k)+b(\partial a/\partial\theta_k)-2c(\partial c/\partial\theta_k)) \begin{bmatrix} bd + ce \\ cd + ae \end{bmatrix}$$

$$+ \frac{1}{(ab-c^2)} \begin{bmatrix} b(\partial d/\partial\theta_k) + d(\partial b/\partial\theta_k) + c(\partial e/\partial\theta_k) + e(\partial c/\partial\theta_k) \\ c(\partial d/\partial\theta_k) + d(\partial c/\partial\theta_k) + a(\partial e/\partial\theta_k) + e(\partial a/\partial\theta_k) \end{bmatrix}$$

Simplifying the defining equation:

Let $f_k = 2[b^2d+bce-ace-c^2d]c_k + [2cbd+c^2e+abe](a_k-b_k)$

$+ [b^2a-c^2b](-2c_kx_k + (b_k-a_k)y_k) + [c(ab-c^2)]((b_k-a_k)x_k + 2c_ky_k)$

$h_k = [-b^2d-bce]a_k+[-ace-c^2d]b_k+[2cbd+c^2e+abe]c_k+[b^2a-c^2b]d_k+[c(ab-c^2)]e_k$

Let $g_k = 2[bcd+c^2e-acd-a^2e]c_k + [2ace+c^2d+abd](a_k-b_k)$

$+ [c(ab-c^2)](-2c_kx_k + (b_k-a_k)y_k) + [a(ab-c^2)]((b_k-a_k)x_k + 2c_ky_k)$

$i_k = [-bcd-c^2e]a_k+[-acd-a^2e]b_k+[2ace+c^2d+abd]C_k+[(ab-c^2)c]d_k+[a(ab-c^2)]e_k$

$$\partial x/\partial\hat{\theta}_k = \{f_k/r_k^2 - \Sigma\, 1/r_j^4 \cdot \partial r_j^2/\partial\hat{\theta}_k \cdot h_j\}/(ab-c^2)^2$$

$$\partial y/\partial\hat{\theta}_k = \{g_k/r_k^2 - \Sigma\, 1/r_j^4 \cdot \partial r_j^2/\partial\hat{\theta}_k \cdot i_j\}/(ab-c^2)^2$$

Let F_k, G_k, H_k, and I_k be f_k, g_k, h_k, and i_k evaluated at the true as in the Taylor Series expansion.

Plugging in for $\partial \hat{r}_i^2/\partial\hat{\theta}_k$ and evaluating at the true yields

$$\partial x/\partial\hat{\theta}_k = \{F_k/R_k^2 - \sum_{j=1}^{n} 2\{((X-x_j)\partial x/\partial\hat{\theta}_k + (Y-y_j)\partial y/\partial\hat{\theta}_k)H_j/R_j^4\}/(AB-C^2)^2$$

$$\partial y/\partial\hat{\theta}_k = \{G_k/R_k^2 - \sum_{j=1}^{n} 2\{((X-x_j)\partial x/\partial\hat{\theta}_k + (Y-y_j)\partial y/\partial\hat{\theta}_k)I_j/R_j^4\}/(AB-C^2)^2$$

where the partials shown are also evaluated at the true.

Let

$$q = [(AB-C^2)^2 + 2\sum_{j=1}^{n}\{(X-x_j)H_j/R_j^4\}][(AB-C^2)^2 + 2\sum_{j=1}^{n}\{(Y-y_j)I_j/R_j^4\}]$$
$$-[2\sum_{j=1}^{n}\{(Y-y_j)H_j/R_j^4\}][2\sum_{j=1}^{n}\{(X-x_j)I_j/R_j^4\}]$$

For partials evaluated at the true

$$\partial x/\partial\hat{\theta}_k = [F_k/R_k^2\{(AB-C^2)^2 + \sum_{j=1}^{n} 2\{((Y-y_j)I_j/R_j^4\} - G_k/R_k^2\{\sum_{j=1}^{n} 2(Y-y_j)H_j/R_j^4\}]/q$$

$$\partial y/\partial\hat{\theta}_k = [-F_k/R_k^2\{\sum_{j=1}^{n} 2\{((X-x_j)I_j/R_j^4\} + G_k/R_k^2\{(AB-C^2)^2 + \sum_{j=1}^{n} 2(X-x_j)H_j/R_j^4\}]/q$$

The answer must be invariant under translations (independent of where the origin is placed.) Fortunately calculations are a good deal easier if we choose $(X,Y)=(0,0)$. (Note we don't do that in the computerized examples however.) With this assumption

$D_k=E_k=0$ for all k. Hence $D=E=0$. Further implying $H_k=I_k=0$ for all k.

Hence $q=(AB-C^2)^4$ and

$$\partial x/\partial\hat{\theta}_k = (AB-C^2)^2F_k/(R_k^2q) = \{B(-2C_kx_k+(B_k-A_k)y_k)+C((B_k-A_k)x_k+2C_ky_k)\}/[R_k^2(AB-C^2)]$$
$$= \{B(-(B_k+A_k)y_k)+C((B_k+A_k)x_k\}/[R_k^2(AB-C^2)]$$
$$= \{-By_k+Cx_k\}/[R_k^2(AB-C^2)]$$

$$\partial y/\partial\hat{\theta}_k = (AB-C^2)^2G_k/(R_k^2q) = \{C(-2C_kx_k+(B_k-A_k)y_k)+A((B_k-A_k)x_k+2C_ky_k)\}/[R_k^2(AB-C^2)]$$
$$= \{-Cy_k+Ax_k\}/[R_k^2(AB-C^2)]$$

Second Partials:

$$\partial^2 r_i^2/\partial\hat{\theta}_k = 2(x-x_i)(\partial^2 x/\partial\hat{\theta}_k^2) + 2(y-y_i)\partial^2 y/\partial\hat{\theta}_k^2 + 2(\partial x/\partial\hat{\theta}_k)^2 + 2(\partial y/\partial\hat{\theta}_k)^2$$

$$\partial^2 a/\partial\hat{\theta}_k^2 = -2(a_k-b_k)/r_k^2 + (4c_k/r_k^4)(\partial r_k^2/\partial\hat{\theta}_k) + \Sigma(2a_i/r_i^6)(\partial r_i^2/\partial\hat{\theta}_k)^2$$
$$+\Sigma(-a_i/r_i^4)(\partial^2 r_i^2/\partial\hat{\theta}_k^2)$$

$$\partial^2 b/\partial\hat{\theta}_k^2 = 2(a_k-b_k)/r_k^2 - (4c_k/r_k^4)(\partial r_k^2/\partial\hat{\theta}_k) + \Sigma(2b_i/r_i^6)(\partial r_i^2/\partial\hat{\theta}_k)^2$$
$$+\Sigma(-b_i/r_i^4)(\partial^2 r_i^2/\partial\hat{\theta}_k^2)$$

$$\partial^2 c/\partial\hat{\theta}_k^2 = -4c_k/r_k^2 - 2((a_k-b_k)/r_k^4)(\partial r_k^2/\partial\hat{\theta}_k) + \Sigma(2c_i/r_i^6)(\partial r_i^2/\partial\hat{\theta}_k)^2$$
$$+\Sigma(-c_i/r_i^4)(\partial^2 r_i^2/\partial\hat{\theta}_k^2)$$

$$\partial^2 d/\partial\hat{\theta}_k^2 = (4c_k y_k - 2(a_k-b_k)x_k)/r_k^2 + 2((1-2b_k)y_k + 2c_k x_k)/r_k^4)(\partial r_k^2/\partial\hat{\theta}_k)$$
$$+ \Sigma(2d_i/r_i^6)(\partial r_i^2/\partial\hat{\theta}_k)^2 + \Sigma(-d_i/r_i^4)(\partial^2 r_i^2/\partial\hat{\theta}_k^2)$$

$$\partial^2 e/\partial\hat{\theta}_k^2 = (4c_k x_k + 2(a_k-b_k)y_k)/r_k^2 + 2((1-2b_k)x_k - 2c_k y_k)/r_k^4)(\partial r_k^2/\partial\hat{\theta}_k)$$
$$+ \Sigma(2e_i/r_i^6)(\partial r_i^2/\partial\hat{\theta}_k)^2 + \Sigma(-e_i/r_i^4)(\partial^2 r_i^2/\partial\hat{\theta}_k^2)$$

$$\begin{matrix}\partial^2 x/\partial\hat{\theta}_k^2 \\ \partial^2 y/\partial\hat{\theta}_k^2\end{matrix} = \frac{1}{(ab-c^2)^3} \quad T1(\partial a/\partial\hat{\theta}_k)^2 + T2(\partial a/\partial\hat{\theta}_k)(\partial b/\partial\hat{\theta}_k) + T3(\partial a/\partial\hat{\theta}_k)(\partial c/\partial\hat{\theta}_k)$$

$$+T4(\partial a/\partial\hat{\theta}_k)(\partial d/\partial\hat{\theta}_k) + T5(\partial a/\partial\hat{\theta}_k)(\partial e/\partial\hat{\theta}_k) + T6(\partial b/\partial\hat{\theta}_k)^2 + T7(\partial b/\partial\hat{\theta}_k)(\partial c/\partial\hat{\theta}_k)$$

$$+T8(\partial b/\partial\hat{\theta}_k)(\partial d/\partial\hat{\theta}_k) + T9(\partial b/\partial\hat{\theta}_k)(\partial e/\partial\hat{\theta}_k) + T10(\partial c/\partial\hat{\theta}_k)^2 + T11(\partial c/\partial\hat{\theta}_k)(\partial d/\partial\hat{\theta}_k)$$

$$+T12(\partial c/\partial\hat{\theta}_k)(\partial e/\partial\hat{\theta}_k) + T13(\partial^2 a/\partial\hat{\theta}_k^2) + T14(\partial^2 b/\partial\hat{\theta}_k^2) + T15(\partial^2 c/\partial\hat{\theta}_k^2)$$

$$+T16(\partial^2 d/\partial\hat{\theta}_k^2) + T17(\partial^2 e/\partial\hat{\theta}_k^2)$$

where T1 through T16 are as listed below.

(no more derivatives will be taken so evaluated derivatives will be used in place of unevaluated ones. In other words, upper-case letters will replace lower-case letters.)

$$T1 = 2B(BD+CE)\begin{bmatrix}B\\C\end{bmatrix}$$

$$T2 = \begin{bmatrix}2ABCE+4C^2BD+2C^2EC\\2ABCD+4C^2AE+2C^2DC\end{bmatrix}$$

$$T3 = \begin{bmatrix}-8B^2CD-6C^2BE-2AB^2E\\-6C^2BD-4ABCE-2AB^2D-4C^2EC\end{bmatrix}$$

$$T4 = (AB-C^2)\begin{bmatrix}-2B^2\\-2BC\end{bmatrix}$$

$$T5 = (AB-C^2)\begin{bmatrix}-2BC\\-2C^2\end{bmatrix}$$

$$T6 = 2A^2\begin{bmatrix}BD+CE\\CD+AE\end{bmatrix} - 2(AB-C^2)\begin{bmatrix}AD\\0\end{bmatrix}$$

$$T7 = -8AC\begin{bmatrix}BD+CE\\CD+AE\end{bmatrix} - 2(AB-C^2)\begin{bmatrix}AE-2CD\\AD\end{bmatrix}$$

$$T8 = (AB-C^2)\begin{bmatrix}-2C^2\\-2AC\end{bmatrix}$$

$$T9 = (AB-C^2)\begin{bmatrix}-2AC\\-2A^2\end{bmatrix}$$

$$T10 = 2C^2\begin{bmatrix}3BD+CE\\2CD+3AE\end{bmatrix} + 2AB\begin{bmatrix}BD+3CE\\3CD+AE\end{bmatrix}$$

$$T11 = (AB-C^2)\begin{bmatrix}4CB\\2AB+C^2\end{bmatrix}$$

$$T12 = (AB-C^2)\begin{bmatrix}2AB+C^2\\4CA\end{bmatrix}$$

$$T13 = (AB-C^2)\begin{bmatrix}-B^2D-BCE\\ -BCD-C^2E\end{bmatrix}$$

$$T14 = (AB-C^2)\begin{bmatrix}-ACE-C^2D\\ -ACD-A^2E\end{bmatrix}$$

$$T15 = (AB-C^2)\begin{bmatrix}2CBD+C^2E+ABE\\ 2ACE+C^2D+ABD\end{bmatrix}$$

$$T16 = (AB-C^2)^2\begin{bmatrix}B\\ C\end{bmatrix}$$

$$T17 = (AB-C^2)^2\begin{bmatrix}C\\ A\end{bmatrix}$$

Again, if it is assumed that $(X,Y)=(0,0)$, then

$T1=T2=T3=T6=T7=T10=T13=T14=T15=0$ (because $D=E=0$)

Infinite Number of LOBs

As in the other methods studied in this report, the first-order bias vector for the Weighted Perpendicular method points along a line segment from the bias for the 2 LOB case to the bias for the infinite LOB case (if there are no significant changes in the configuration). First-order bias as sample size approaches infinity is as follows.

$$\text{Asymptotic first-order bias} = \frac{\begin{bmatrix}B & C\\ C & A\end{bmatrix}\Sigma\begin{bmatrix}x_i\\ y_i\end{bmatrix}\cdot\sigma_i^2/r_i^2}{AB-C^2}$$

It is possible to set $C=0$ by finding the "average angle" of the angles between LOBs and true north. Along this "average angle", asymptotic first-order bias will lie short.

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